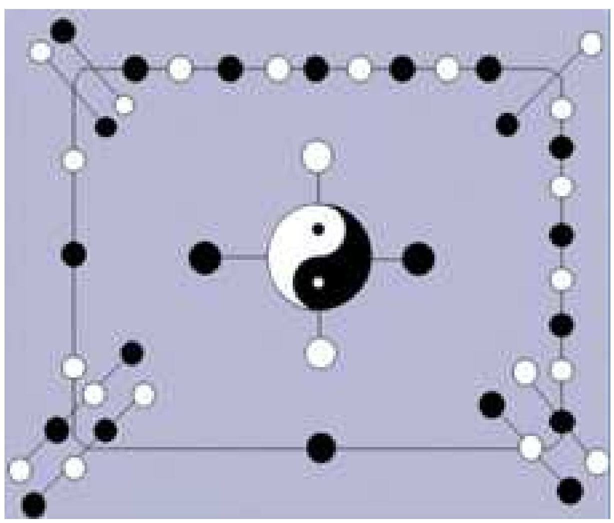
ISBN 1-59973-076-6

VOLUME 3, 2008

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



October, 2008

Vol.3, 2008 ISBN 1-59973-076-6

Mathematical Combinatorics

(International Book Series)

Edited By Linfan MAO

October, 2008

Aims and Scope: The **Mathematical Combinatorics (International Book Series)** (*ISBN 1-59973-076-6*) is a fully refereed international book series and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces,$\cdots$, etc.. Smarandache geometries;

Differential Geometry; Geometry on manifolds;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics;

Mathematical theory on gravitational fields; Mathematical theory on parallel universes;

Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available in **microfilm format** and can be ordered online from:

Books on Demand
ProQuest Information & Learning
300 North Zeeb Road
P.O.Box 1346, Ann Arbor
MI 48106-1346, USA
Tel:1-800-521-0600(Customer Service)
URL: http://madisl.iss.ac.cn/IJMC.htm/

Indexing and Reviews: Mathematical Reviews(USA), Zentralblatt fur Mathematik(Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Computing Review (USA), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by a mail or an email directly to the Editor-in-Chief.

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science
Beijing, 100080, P.R.China
Email: maolinfan@163.com

Printed in the United States of America
Price: US$48.00

New opinions are always suspected, and usually opposed, without any other reason but because they are not already common.

John Locke, a British Philosopher.

Math.Combin.Book.Ser. Vol.3 (2008), 01-27

Extending Homomorphism Theorem to Multi-Systems

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract: The multi-laterality of WORLD implies multi-systems to be its best candidate model for ones cognition on nature, which is also included in an ancient book of China, *TAO TEH KING* written by Lao Zi, an ancient philosopher of China. Then *how it works to mathematics, not suspended in thought*? This paper explains this action by mathematical logic on mathematical systems generalized to Smarandache systems, or such systems with combinatorial structures, i.e., combinatorial systems, and shows how to extend the homomorphism theorem in abstract algebra to multi-systems or combinatorial systems. All works in this paper are motivated by a combinatorial speculation of mine which is reformed on combinatorial systems and can be also applied to geometry.

Key Words: Homomorphism theorem, multi-system, combinatorial system.

AMS(2000): 05E15, 08A02, 15A03, 20E07, 51M15.

§1. Introduction

The WORLD is a multi-lateral one. The entirely realization on WORLD is very difficult for the limitation of mankind on the earth. In Fig.1.1, it is shown part of the WORLD by eyes of a man on the earth.

Fig.1.1

[1]Received April 10, 2008. Accepted August 2, 2008.

[2]Reported at the *4th International Conference on Number Theory and Smarandache Problems*, March 22-24, 2008, Shanxi Xiangyang, P.R.China.

The multi-laterality of the WORLD implies multi-systems to be its best candidate model for ones cognition on nature. This is also included in a well-known Chinese ancient book *TAO TEH KING* written by *LAO ZI*. In this book, only with nearly 5000 words, we can find many sentences for cognition of our world, such as those of the following (see [5] for details), each of them is explained by a concrete figure.

SENTENCE 1. *All things that we can acknowledge is determined by our eyes, or ears, or nose, or tongue, or body or passions, i.e., these six organs.* Such as those shown in Fig.1.2.

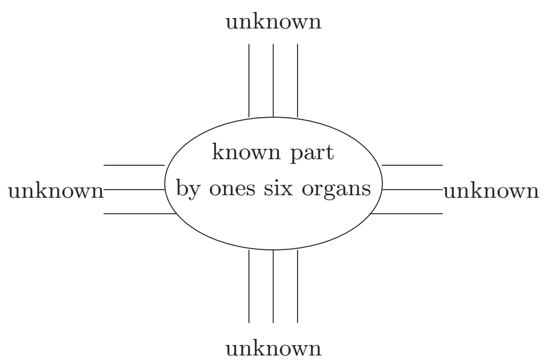


Fig.1.2

SENTENCE 2. *The Tao gives birth to One. One gives birth to Two. Two gives birth to Three. Three gives birth to all things. All things have their backs to the female and stand facing the male. When male and female combine, all things achieve harmony.* Shown in Fig.1.3.

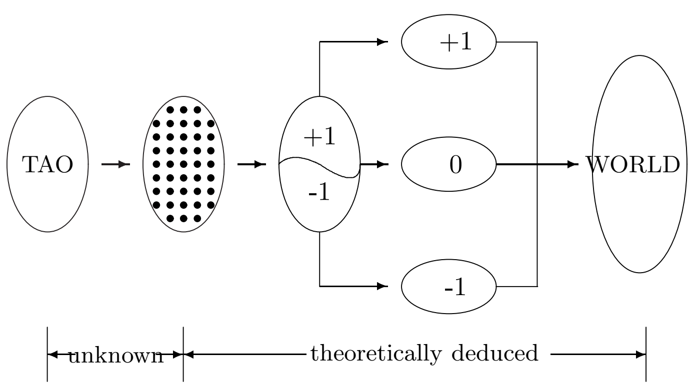


Fig.1.3

SENTENCE 3. *Mankind follows the earth. Earth follows the universe. The universe follows the Tao. The Tao follows only itself.* Such as those shown in Fig.1.4

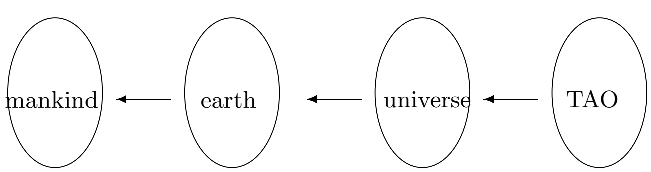


Fig.1.4

SENTENCE 4. Have *and* Not have *exist jointly ahead of the birth of the earth and the sky.* This means that any thing have two sides. One is the positive. Another is the negative. We can not say a thing existing or not just by our six organs because its existence independent on our living.

What can we learn from these words? How can we apply them in mathematics of the 21st century? All these sentences mean that our world is a multi-one. For characterizing its behavior, We should construct a multi-system model for the WORLD, also called parallel universes ([23]), such as those shown in Fig.1.5.

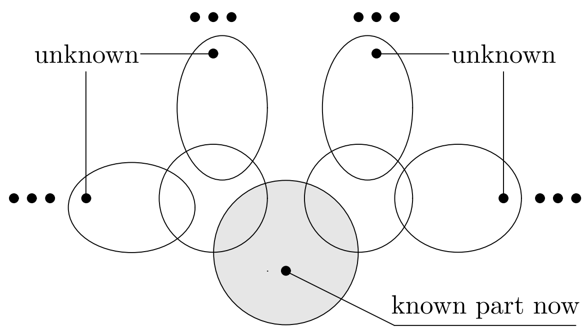


Fig.1.5

Whence, it is a Smarandache multi-system with a combinatorial structure G, i.e., a combinatorial system $\mathscr{C}_G$.

In this paper, we will characterize such systems, and generalize the well-known homomorphism theorem in group theory to multi-systems, particularly, to multi-groups, multi-rings and multi-modules (see [11] – [14] for details). In the remain part of this section, we recall some terminologies in mathematical logic and define what is a mathematical system. These Smarandache systems and combinatorial systems are introduced in Section 2. After that, we show how to generalize the homomorphism theorem of groups to multi-systems in the following sections. Terminologies and notations not defined here follow the reference [1], [18], [24] for algebra, [2], [3] and [7] – [9] for graphs.

A *proposition* p on a set Σ is a declarative sentence on elements in Σ that is either true or false but not both. The statements *it is not the case that* p and *it is the opposite case that* p are still propositions, called the *negation* or *anti-proposition* of p, denoted by non-p or anti-p, respectively. Generally, $\text{non} - p \neq \text{anti} - p$. The structure of anti-$p$ is very clear, but non-p is not. An oppositive or negation of a proposition are shown in Fig.1.6(1) and (2).

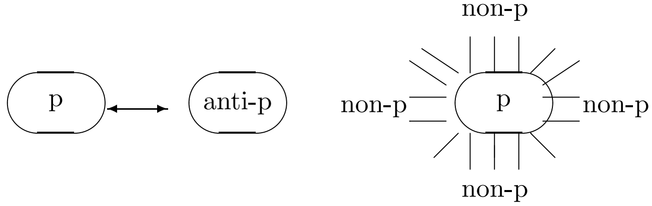


Fig.1.6

A proposition and its non-proposition jointly exist in the world. Its truth or false can be only decided by logic inference, independent on one knowing it or not.

A norm inference is called implication. An *implication* $p \to q$, *i.e., if* p *then* q, is a proposition that is false when p is true but q false and true otherwise. There are three propositions related with $p \to q$, namely, $q \to p$, $\neg q \to \neg p$ and $\neg p \to \neg q$, called the *converse*, *contrapositive* and *inverse* of $p \to q$. Two propositions are called *equivalent* if they have the same truth value. It can be shown immediately that *an implication and its contrapositive are equivalent*. This fact is commonly used in mathematical proofs, i.e., we can either prove the proposition $p \to q$ or $\neg q \to \neg p$ in the proof of $p \to q$, not the both.

A *rule* on a set Σ is a mapping

$$\underbrace{\Sigma \times \Sigma \cdots \times \Sigma}_{n} \to \Sigma$$

for some integers n. A *mathematical system* is a pair $(\Sigma; \mathcal{R})$, where Σ is a set consisting mathematical objects, infinite or finite and $\mathcal{R}$ is a collection of rules on Σ by logic providing all these resultants are still in Σ, i.e., elements in Σ is closed under rules in $\mathcal{R}$.

Two mathematical systems $(\Sigma_1; \mathcal{R}_1)$ and $(\Sigma_2; \mathcal{R}_2)$ are *isomorphic* if there is a $1-1$ mapping $\omega : \Sigma_1 \to \Sigma_2$ such that for elements $a, b, \cdots, c \in \Sigma_1$,

$$\omega(\mathcal{R}_1(a, b, \cdots, c)) = \mathcal{R}_2(\omega(a), \omega(b), \cdots, \omega(c)) \in \Sigma_2.$$

Generally, we do not distinguish isomorphic systems in mathematics. Examples for mathematical systems are shown in the following.

Example 1.1 A *group* $(G; \circ)$ in classical algebra is a mathematical system $(\Sigma_G; \mathcal{R}_G)$, where $\Sigma_G = G$ and

$$\mathcal{R}_G = \{R_1^G; R_2^G, R_3^G\},$$

with

R_1^G: $(x \circ y) \circ z = x \circ (y \circ z)$ for $\forall x, y, z \in G$;

R_2^G: there is an element $1_G \in G$ such that $x \circ 1_G = x$ for $\forall x \in G$;

R_3^G: for $\forall x \in G$, there is an element $y, y \in G$, such that $x \circ y = 1_G$.

Then, the well-known homomorphism theorem of groups is restated in the next.

Homomorphism Theorem *Let $\sigma : G \to G'$ be a homomorphism from groups G to G'. Then*

$$G/\mathrm{Ker}\sigma \;\cong\; G'.$$

$\square$

Example 1.2 A ring $(R; +, \circ)$ with two binary closed operations $+, \circ$is a mathematical system $(\Sigma; \mathcal{R})$, where $\Sigma = R$ and $\mathcal{R} = \{R_1; R_2, R_3, R_4\}$ with

R_1: $x + y, x \circ y \in R$ for $\forall x, y \in R$;

R_2: $(R; +)$ is a commutative group, i.e., $x + y = y + x$ for $\forall x, y \in R$;

R_3: $(R; \circ)$ is a semigroup;

R_4: $x \circ (y+z) = x \circ y + x \circ z$ and $(x+y) \circ z = x \circ z + y \circ z$ for $\forall x, y, z \in R$.

Example 1.3 An Euclidean geometry on the plane $\mathbf{R}^2$ is a a mathematical system $(\Sigma_E; \mathcal{R}_E)$, where $\Sigma_E = \{\text{points and lines on } \mathbf{R}^2\}$ and $\mathcal{R}_E = \{\text{Hilbert's 21 axioms on Euclidean geometry}\}$.

A mathematical $(\Sigma; \mathcal{R})$ can be constructed dependent on the set Σ or on rules $\mathcal{R}$. The former requires each rule in $\mathcal{R}$ closed in Σ. But the later requires that $\mathcal{R}(a, b, \cdots, c)$ in the final set $\widehat{\Sigma}$, which means that $\widehat{\Sigma}$ maybe an extended of the set Σ. In this case, we say $\widehat{\Sigma}$ is generated by Σ under rules $\mathcal{R}$, denoted by $\langle \Sigma; \mathcal{R} \rangle$.

§2. Combinatorial System

By the view of *LAO ZHI* in Section 1, we should construct such mathematical systems $(\Sigma; \mathcal{R})$ for the WORLD in which a proposition with its non-proposition validated turn up in the set Σ, or invalidated but in multiple ways in Σ, which is a Smarandache system defined in the next.

Definition 2.1 *A rule in a mathematical system $(\Sigma; \mathcal{R})$ is said to be Smarandachely denied if it behaves in at least two different ways within the same set Σ, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache system $(\Sigma; \mathcal{R})$ is a mathematical system which has at least one Smarandachely denied rule in $\mathcal{R}$.

Definition 2.2 *For an integer $m \geq 2$, let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be m mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma}; \widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad \textit{and} \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Certainly, we can construct Smarandache systems by applying Smarandache multi-spaces, particularly, Smarandache geometries ([4], [7]-[17]).

These Smarandache systems $(\Sigma; \mathcal{R})$ defined in Definition 2.1 consider the behavior of a proposition and its non-proposition in the same set Σ without distinguishing the guises of these non-propositions. In fact, there are many appearing ways for non-propositions of a proposition in Σ. For describing their behavior, the combinatorial systems are needed.

Definition 2.3 *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1; \mathcal{R}_1)$,$(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ for an integer m, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

Unless its combinatorial G structure, these cardinalities $|\Sigma_i \bigcap \Sigma_j|$, called the *coupling constants* in a combinatorial system $\mathscr{C}_G$ also determine its structure if $\Sigma_i \bigcap \Sigma_j \neq \emptyset$ for integers $1 \leq i, j \leq m$. For emphasizing its coupling constants, we denote a combinatorial system $\mathscr{C}_G$ by $\mathscr{C}_G(l_{ij}, 1 \leq i, j \leq m)$ if $l_{ij} = |\Sigma_i \bigcap \Sigma_j| \neq 0$.

Let $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ be two combinatorial systems with

$$\mathscr{C}_G^{(1)} = (\bigcup_{i=1}^{m} \Sigma_i^{(1)};\ \bigcup_{i=1}^{m} \mathcal{R}_i^{(1)}), \qquad \mathscr{C}_G^{(2)} = (\bigcup_{i=1}^{n} \Sigma_i^{(2)};\ \bigcup_{i=1}^{n} \mathcal{R}_i^{(2)}).$$

A *homomorphism* $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a mapping $\varpi : \bigcup\limits_{i=1}^{m} \Sigma_i^{(1)} \to \bigcup\limits_{i=1}^{n} \Sigma_i^{(2)}$ and $\varpi : \bigcup\limits_{i=1}^{m} \mathcal{R}_i^{(1)}) \to \bigcup\limits_{i=1}^{n} \mathcal{R}_i^{(2)}$ such that

$$\varpi|_{\Sigma_i}(a\mathcal{R}_i^{(1)}b) = \varpi|_{\Sigma_i}(a)\varpi|_{\Sigma_i}(\mathcal{R}_i^{(1)})\varpi|_{\Sigma_i}(b)$$

for $\forall a, b \in \Sigma_i^{(1)}$, $1 \leq i \leq m$, where $\varpi|_{\Sigma_i}$ denotes the constraint mapping of ϖ on the mathematical system $(\Sigma_i, \mathcal{R}_i)$. Further more, if $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a $1-1$ mapping, then we say these $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ are *isomorphic* with an isomorphism ϖ between them.

A homomorphism $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ naturally induces a mappings $\varpi|_G$ on the graph G_1 and G_2 by

$$\varpi|_G : V(G_1) \to \varpi(V(G_1)) \subset V(G_2) \text{ and}$$

$$\varpi|_G : (\Sigma_i, \Sigma_j) \in E(G_1) \to (\varpi(\Sigma_i), \varpi(\Sigma_j)) \in E(G_2), 1 \leq i, j \leq m.$$

With these notations, a criterion for isomorphic combinatorial systems is presented in the following.

Theorem 2.1 *Two combinatorial systems $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$ are isomorphic if and only if there is a $1-1$ mapping $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ such that*

(i) $\varpi|_{\Sigma_i^{(1)}}$ *is an isomorphism and* $\varpi|_{\Sigma_i^{(1)}}(x) = \varpi|_{\Sigma_j^{(1)}}(x)$ *for* $\forall x \in \Sigma_i^{(1)} \cap \Sigma_j^{(1)}$, $1 \leq i, j \leq m$;

(ii) $\varpi|_G : G_1 \to G_2$ *is an isomorphism.*

Proof If $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is an isomorphism, considering the constraint mappings of ϖ on the mathematical system $(\Sigma_i, \mathcal{R}_i)$ for an integer $i, 1 \leq i \leq m$ and the graph $G_1^{(1)}$, then we find isomorphisms $\varpi|_{\Sigma_i^{(1)}}$ and $\varpi|_G$.

Conversely, if these isomorphism $\varpi|_{\Sigma_i^{(1)}}, 1 \leq i \leq m$ and $\varpi|_G$ exist, we can construct a mapping $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ by

$$\varpi(a) = \varpi|_{\Sigma_1}(a) \ \text{ if } \ a \in \Sigma_i \ \text{ and } \ \varpi(\circ) = \varpi|_{\Sigma_1}(\circ) \ \text{ if } \ \circ \in \mathcal{R}_i, 1 \leq i \leq m.$$

Then we know that

$$\varpi|_{\Sigma_i}(a\mathcal{R}_i^{(1)}b) = \varpi|_{\Sigma_i}(a)\varpi|_{\Sigma_i}(\mathcal{R}_i^{(1)})\varpi|_{\Sigma_i}(b)$$

for $\forall a, b \in \Sigma_i^{(1)}$, $1 \leq i \leq m$ by definition. Whence, $\varpi : \mathscr{C}_G^{(1)} \to \mathscr{C}_G^{(2)}$ is a homomorphism. Similarly, we can know that $\varpi^{-1} : \mathscr{C}_G^{(2)} \to \mathscr{C}_G^{(1)}$ is also an homomorphism. Therefore, ϖ is an

isomorphism between $\mathscr{C}_G^{(1)}$ and $\mathscr{C}_G^{(2)}$. $\square$

For understanding well the multiple behavior of the WORLD, namely, its overlap and hybrid, a combinatorial system should be constructed. Then *what is its relation with classical mathematical sciences*? *What is its developing way for mathematical sciences*? I have presented an idea of combinatorial notion in Chapter 5 of [7], then stated formally as the *combinatorial conjecture for mathematics* in [10] and [16], the last was reported at *the 2nd Conference on Combinatorics and Graph Theory of China* in 2006.

Combinatorial Conjecture *Any mathematical system* $(\Sigma;\mathcal{R})$ *is a combinatorial system* $\mathscr{C}_G(l_{ij}, 1 \leq i,j \leq m)$.

This conjecture is not just like an open problem, but more like a deeply thought, which opens a entirely way for advancing the modern mathematics. Here, we need further clarification for this conjecture. In fact, it indeed means combinatorial notions following for researchers.

(1) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(2) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(3) One can make one combination of different branches in mathematics and find new results after then.

Whence, a mathematical system can not be ended if it has not been combinatorialization and all mathematical systems can not be ended if its combinatorialization has not completed yet. The reader can applies this combinatorial notion to all of his research work, and then finds his combinatorial fields.

§3. Algebraic Systems

Let $(\mathscr{A};\circ)$ be an algebraic system and $\mathscr{B} \subset \mathscr{A}$, if $(\mathscr{B};\circ)$ is still an algebraic system, then we call it an *algebraic sub-system of* $(\mathscr{A};\circ)$, denoted by $\mathscr{B} \prec \mathscr{A}$. Similarly, an algebraic sub-system is called a *subgroup* if it is group itself.

Let $(\mathscr{A};\circ)$ be an algebraic system and $\mathscr{B} \prec \mathscr{A}$. For $\forall a \in \mathscr{A}$, define a coset $a \circ \mathscr{B}$ of $\mathscr{B}$ in $\mathscr{A}$ by

$$a \circ \mathscr{B} = \{a \circ b | \forall b \in \mathscr{B}\}.$$

Define a *quotient set* $\mathfrak{S} = \mathscr{A}/\mathscr{B}$ consists of all cosets of $\mathscr{B}$ in $\mathscr{A}$ and let R be a minimal set with $\mathfrak{S} = \{r \circ \mathscr{B} | r \in R\}$, called a *representation of* $\mathfrak{S}$. Then

Theorem 3.1 *If* $(\mathscr{B};\circ)$ *is a subgroup of an associative system* $(\mathscr{A};\circ)$, *then*

(*i*) *for* $\forall a, b \in \mathscr{A}$, $(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) = \emptyset$ *or* $a \circ \mathscr{B} = b \circ \mathscr{B}$, *i.e.,* $\mathfrak{S}$ *is a partition of* $\mathscr{A}$;

(*ii*) *define an operation* $\bullet$ *on* $\mathfrak{S}$ *by*

$$(a \circ \mathscr{B}) \bullet (b \circ \mathscr{B}) = (a \circ b) \circ \mathscr{B},$$

then $(\mathfrak{S}; \bullet)$ *is an associative algebraic system, called a quotient system of* $\mathscr{A}$ *to* $\mathscr{B}$*. Particularly, if there is a representation* R *whose each element has an inverse in* $(\mathscr{A}; \circ)$ *with unit* $1_{\mathscr{A}}$*, then* $(\mathfrak{S}; \bullet)$ *is a group, called a quotient group of* $\mathscr{A}$ *to* $\mathscr{B}$*.*

Proof For (i), notice that if

$$(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) \neq \emptyset$$

for $a, b \in \mathscr{A}$, then there are elements $c_1, c_2 \in \mathscr{B}$ such that $a \circ c_1 = b \circ c_2$. By assumption, $(\mathscr{B}; \circ)$ is a subgroup of $(\mathscr{A}; \circ)$, we know that there exists an inverse element $c_1^{-1} \in \mathscr{B}$, i.e., $a = b \circ c_2 \circ c_1^{-1}$. Therefore, we get that

$$\begin{aligned} a \circ \mathscr{B} &= (b \circ c_2 \circ c_1^{-1}) \circ \mathscr{B} \\ &= \{(b \circ c_2 \circ c_1^{-1}) \circ c | \forall c \in \mathscr{B}\} \\ &= \{b \circ c | \forall c \in \mathscr{B}\} \\ &= b \circ \mathscr{B} \end{aligned}$$

by the associative law and $(\mathscr{B}; \circ)$ is a group gain, i.e., $(a \circ \mathscr{B}) \cap (b \circ \mathscr{B}) = \emptyset$ or $a \circ \mathscr{B} = b \circ \mathscr{B}$.

By definition of $\bullet$ on $\mathfrak{S}$ and (i), we know that $(\mathfrak{S}; \bullet)$ is an algebraic system. For $\forall a, b, c \in \mathscr{A}$, by the associative laws in $(\mathscr{A}; \circ)$, we find that

$$\begin{aligned} ((a \circ \mathscr{B}) \bullet (b \circ \mathscr{B})) \bullet (c \circ \mathscr{B}) &= ((a \circ b) \circ \mathscr{B}) \bullet (c \circ \mathscr{B}) \\ &= ((a \circ b) \circ c) \circ \mathscr{B} = (a \circ (b \circ c)) \circ \mathscr{B} \\ &= (a \circ \mathscr{B}) \circ ((b \circ c) \circ \mathscr{B}) \\ &= (a \circ \mathscr{B}) \bullet ((b \circ \mathscr{B}) \bullet (c \circ \mathscr{B})). \end{aligned}$$

Now if there is a representation R whose each element has an inverse in $(\mathscr{A}; \circ)$ with unit $1_{\mathscr{A}}$, then it is easy to know that $1_{\mathscr{A}} \circ \mathscr{B}$ is the unit and $a^{-1} \circ \mathscr{B}$ the inverse element of $a \circ \mathscr{B}$ in $\mathfrak{S}$. Whence, $(\mathfrak{S}; \bullet)$ is a group. □

Corollary 3.1 *For a subgroup* $(\mathscr{B}; \circ)$ *of a group* $(\mathscr{A}; \circ)$*,* $(\mathfrak{S}; \bullet)$ *is a group.*

Corollary 3.2(Lagrange theorem) *For a subgroup* $(\mathscr{B}; \circ)$ *of a group* $(\mathscr{A}; \circ)$*,*

$$|\mathscr{B}| \mid |\mathscr{A}|.$$

Proof Since $a \circ c_1 = a \circ c_2$ implies that $c_1 = c_2$ in this case, we know that

$$|a \circ \mathscr{B}| = |\mathscr{B}|$$

for $\forall a \in \mathscr{A}$. Applying Theorem 2.2.4$(i)$, we find that

$$|\mathscr{A}| = \sum_{r \in R} |r \circ \mathscr{B}| = |R||\mathscr{B}|,$$

for a representation R, i.e., $|\mathscr{B}| \mid |\mathscr{A}|$. □

Although the operation $\bullet$ in $\mathfrak{S}$ is introduced by the operation $\circ$ in $\mathscr{A}$, it may be $\bullet \neq \circ$. Now if $\bullet = \circ$, i.e.,

$$(a \circ \mathscr{B}) \circ (b \circ \mathscr{B}) = (a \circ b) \circ \mathscr{B}, \qquad (3-1)$$

the subgroup $(\mathscr{B}; \circ)$ is called a *normal subgroup of* $(\mathscr{B}; \circ)$, denoted by $\mathscr{B} \trianglelefteq \mathscr{A}$. In this case, if there exist inverses of a, b, we know that

$$\mathscr{B} \circ b \circ \mathscr{B} = b \circ \mathscr{B}$$

by product a^{-1} from the left on both side of $(3-1)$. Now since $(\mathscr{B}; \circ)$ is a subgroup, we get that

$$b^{-1} \circ \mathscr{B} \circ b = \mathscr{B},$$

which is the usually definition for a normal subgroup of a group. Certainly, we can also get

$$b \circ \mathscr{B} = \mathscr{B} \circ b$$

by this equality, which can be used to define a *normal subgroup of a algebraic system* with or without inverse element for an element in this system.

Now let $\varpi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ be a homomorphism from an algebraic system $(\mathscr{A}_1; \circ_1)$ with unit $1_{\mathscr{A}_1}$ to $(\mathscr{A}_2; \circ_2)$ with unit $1_{\mathscr{A}_2}$. Define the *inverse set* $\varpi^{-1}(a_2)$ *for an element* $a_2 \in \mathscr{A}_2$ by

$$\varpi^{-1}(a_2) = \{a_1 \in \mathscr{A}_1 | \varpi(a_1) = a_2\}.$$

Particularly, if $a_2 = 1_{\mathscr{A}_2}$, the inverse set $\varpi^{-1}(1_{\mathscr{A}_2})$ is important in algebra and called the *kernel of* ϖ and denoted by $\mathrm{Ker}(\varpi)$, which is a normal subgroup of $(\mathscr{A}_1; \circ_1)$ if it is associative and each element in $\mathrm{Ker}(\varpi)$ has inverse element in $(\mathscr{A}_1; \circ_1)$. In fact, by definition, for $\forall a, b, c \in \mathscr{A}_1$, we know that

(1) $(a \circ b) \circ c = a \circ (b \circ c) \in \mathrm{Ker}(\varpi)$ for $\varpi((a \circ b) \circ c) = \varpi(a \circ (b \circ c)) = 1_{\mathscr{A}_2}$;

(2) $1_{\mathscr{A}_2} \in \mathrm{Ker}(\varpi)$ for $\varpi(1_{\mathscr{A}_1}) = 1_{\mathscr{A}_2}$;

(3) $a^{-1} \in \mathrm{Ker}(\varpi)$ for $\forall a \in \mathrm{Ker}(\varpi)$ if a^{-1} exists in $(\mathscr{A}_1; \circ_1)$ since $\varpi(a^{-1}) = \varpi^{-1}(a) = 1_{\mathscr{A}_2}$;

(4) $a \circ \mathrm{Ker}(\varpi) = \mathrm{Ker}(\varpi) \circ a$ for

$$\varpi(a \circ \mathrm{Ker}(\varpi)) = \varpi(\mathrm{Ker}(\varpi) \circ a) = \varpi^{-1}(\varpi(a))$$

by definition. Whence, $\mathrm{Ker}(\varpi)$ is a normal subgroup of $(\mathscr{A}_1; \circ_1)$.

Theorem 3.2 *Let* $\varpi : \mathscr{A}_1 \rightarrow \mathscr{A}_2$ *be an onto homomorphism from associative systems* $(\mathscr{A}_1; \circ_1)$ *to* $(\mathscr{A}_2; \circ_2)$ *with units* $1_{\mathscr{A}_1}$, $1_{\mathscr{A}_2}$. *Then*

$$\mathscr{A}_1/\mathrm{Ker}(\varpi) \;\cong\; (\mathscr{A}_2;\circ_2)$$

if each element of $\mathrm{Ker}(\varpi)$ *has an inverse in* $(\mathscr{A}_1;\circ_1)$.

Proof We have known that $\mathrm{Ker}(\varpi)$ is a subgroup of $(\mathscr{A}_1;\circ_1)$. Whence $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ is a quotient system. Define a mapping $\varsigma : \mathscr{A}_1/\mathrm{Ker}(\varpi) \to \mathscr{A}_2$ by

$$\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) = \varpi(a).$$

We prove this mapping is an isomorphism. Notice that ς is onto by that ϖ is an onto homomorphism. Now if $a \circ_1 \mathrm{Ker}(\varpi) \neq b \circ_1 \mathrm{Ker}(\varpi)$, then $\varpi(a) \neq \varpi(b)$. Otherwise, we find that $a \circ_1 \mathrm{Ker}(\varpi) = b \circ_1 \mathrm{Ker}(\varpi)$, a contradiction. Whence, $\varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \neq \varsigma(b \circ_1 \mathrm{Ker}(\varpi))$, i.e., ς is a bijection from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $\mathscr{A}_2$.

Since ϖ is a homomorphism, we get that

$$\begin{aligned}&\varsigma((a \circ_1 \mathrm{Ker}(\varpi)) \circ_1 (b \circ_1 \mathrm{Ker}(\varpi)))\\ &= \varsigma(a \circ_1 \mathrm{Ker}(\varpi)) \circ_2 \varsigma(b \circ_1 \mathrm{Ker}(\varpi))\\ &= \varpi(a) \circ_2 \varpi(b),\end{aligned}$$

i.e., ς is an isomorphism from $\mathscr{A}_1/\mathrm{Ker}(\varpi)$ to $(\mathscr{A}_2;\circ_2)$. □

Corollary 3.3 *Let* $\varpi : \mathscr{A}_1 \to \mathscr{A}_2$ *be an onto homomorphism from groups* $(\mathscr{A}_1;\circ_1)$ *to* $(\mathscr{A}_2;\circ_2)$. *Then*

$$\mathscr{A}_1/\mathrm{Ker}(\varpi) \;\cong\; (\mathscr{A}_2;\circ_2).$$

§4. Multi-Operation Systems

A *multi-operation system* is a pair $(\mathscr{H};\widetilde{O})$ with a set $\mathscr{H}$ and an operation set

$$\widetilde{O} = \{\circ_i \mid 1 \leq i \leq l\}$$

on $\mathscr{H}$ such that each pair $(\mathscr{H};\circ_i)$ is an algebraic system. A multi-operation system $(\mathscr{H};\widetilde{O})$ is *associative* if for $\forall a, b, c \in \mathscr{H}$, $\forall \circ_1, \circ_2 \in \widetilde{O}$, there is

$$(a \circ_1 b) \circ_2 c = a \circ_1 (b \circ_2 c).$$

Such a system is called an *associative multi-operation system*. A multi-operation system $(\mathscr{H};\widetilde{O})$ is *distributive* if $\widetilde{O} = \mathcal{O}_1 \cup \mathcal{O}_1$ with $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ such that

$$a \circ_1 (b \circ_2 c) = (a \circ_1 b) \circ_2 (a \circ_1 c) \text{ and } (b \circ_2 c) \circ_1 a = (b \circ_1 a) \circ_2 (c \circ_1 a)$$

for $\forall a, b, c \in \mathscr{H}$ and $\forall \circ_1 \in \mathcal{O}_1$, $\circ_2 \in \mathcal{O}_2$. Denoted such a system by $(\mathscr{H};\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$.

Let $(\mathscr{H},\widetilde{O})$ be a multi-operation system and $\mathscr{G} \subset \mathscr{H}$, $\widetilde{Q} \subset \widetilde{O}$. If $(\mathscr{G};\widetilde{Q})$ is itself a multi-operation system, we call $(\mathscr{G};\widetilde{Q})$ a *multi-operation subsystem of* $(\mathscr{H},\widetilde{O})$*)*, denoted by $(\mathscr{G};\widetilde{Q}) \prec (\mathscr{H},\widetilde{O})$. In those of subsystems, the $(\mathscr{G};\widetilde{O})$ is taking over an important position in the following.

Assume $(\mathscr{G};\widetilde{O}) \prec (\mathscr{H},\widetilde{O})$. For $\forall a \in \mathscr{H}$ and $\circ_i \in \widetilde{O}$, where $1 \leq i \leq l$, define a coset $a \circ_i \mathscr{G}$ by

$$a \circ_i \mathscr{G} = \{a \circ_i b| \text{ for } \forall b \in \mathscr{G}\},$$

and let

$$\mathscr{H} = \bigcup_{a \in R, \circ \in \widetilde{P} \subset \widetilde{O}} a \circ \mathscr{G}.$$

Then the set

$$\mathscr{Q} = \{a \circ \mathscr{G} | a \in R, \circ \in \widetilde{P} \subset \widetilde{O}\}$$

is called a *quotient set of* $\mathscr{G}$ *in* $\mathscr{H}$ *with a representation pair* $(R, \widetilde{P})$, denoted by $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$.

Two multi-operation systems $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ are called *homomorphic* if there is a mapping $\omega : \mathscr{H}_1 \to \mathscr{H}_2$ with $\omega : \widetilde{O}_1 \to \widetilde{O}_2$ such that for $a_1, b_1 \in \mathscr{H}_1$ and $\circ_1 \in \widetilde{O}_1$, there exists an operation $\circ_2 = \omega(\circ_1) \in \widetilde{O}_2$ enables that

$$\omega(a_1 \circ_1 b_1) = \omega(a_1) \circ_2 \omega(b_1).$$

Similarly, if ω is a bijection, $(\mathscr{H}_1;\widetilde{O}_1)$ and $(\mathscr{H}_2;\widetilde{O}_2)$ are called *isomorphic*, and if $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, ω is called an *automorphism on* $\mathscr{H}$.

Theorem 4.1 *Let* $(\mathscr{H},\widetilde{O})$ *be an associative multi-operation system with a unit* $1_\circ$ *for* $\forall \circ \in \widetilde{O}$ *and* $\mathscr{G} \subset \mathscr{H}$.

(i) If $\mathscr{G}$ *is closed for operations in* $\widetilde{O}$ *and for* $\forall a \in \mathscr{G}, \circ \in \widetilde{O}$, *there exists an inverse element* $a_\circ^{-1}$ *in* $(\mathscr{G};\circ)$, *then there is a representation pair* $(R,\widetilde{P})$ *such that the quotient set* $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ *is a partition of* $\mathscr{H}$, *i.e., for* $a, b \in \mathscr{H}, \forall \circ_1, \circ_2 \in \widetilde{O}$, $(a \circ_1 \mathscr{G}) \cap (b \circ_2 \mathscr{G}) = \emptyset$ *or* $a \circ_1 \mathscr{G} = b \circ_2 \mathscr{G}$.

(ii) For $\forall \circ \in \widetilde{O}$, *define an operation* $\circ$ *on* $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ *by*

$$(a \circ_1 \mathscr{G}) \circ (b \circ_2 \mathscr{G}) = (a \circ b) \circ_1 \mathscr{G}.$$

Then $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ *is an associative multi-operation system. Particularly, if there is a representation pair* $(R,\widetilde{P})$ *such that for* $\circ' \in \widetilde{P}$, *any element in* R *has an inverse in* $(\mathscr{H};\circ')$, *then* $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$ *is a group.*

Proof For $a, b \in \mathscr{H}$, if there are operations $\circ_1, \circ_2 \in \widetilde{O}$ with $(a \circ_1 \mathscr{G}) \cap (b \circ_2 \mathscr{G}) \neq \emptyset$, then there must exist $g_1, g_2 \in \mathscr{G}$ such that $a \circ_1 g_1 = b \circ_2 g_2$. By assumption, there is an inverse element c_1^{-1} in the system $(\mathscr{G};\circ_1)$. We find that

$$\begin{aligned} a \circ_1 \mathscr{G} &= (b \circ_2 g_2 \circ_1 c_1^{-1}) \circ_1 \mathscr{G} \\ &= b \circ_2 (g_2 \circ_1 c_1^{-1} \circ_1 \mathscr{G}) = b \circ_2 \mathscr{G} \end{aligned}$$

by the associative law. This implies that $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is a partition of $\mathscr{H}$.

Notice that $\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})}$ is closed under operations in $\widetilde{P}$ by definition. It is a multi-operation system. For $\forall a,b,c \in R$ and operations $\circ_1, \circ_2, \circ_3, \circ^1, \circ^2 \in \widetilde{P}$ we know that

$$\begin{aligned}((a\circ_1 \mathscr{G})\circ^1 (b\circ_2 \mathscr{G}))\circ^2 (c\circ_3 \mathscr{G}) &= ((a\circ^1 b)\circ_1 \mathscr{G})\circ^2 (c\circ_3 \mathscr{G})\\ &= ((a\circ^1 b)\circ^2 c)\circ_1 \mathscr{G}\end{aligned}$$

and

$$\begin{aligned}(a\circ_1 \mathscr{G})\circ^1 ((b\circ_2 \mathscr{G})\circ^2 (c\circ_3 \mathscr{G})) &= (a\circ_1 \mathscr{G})\circ_1 ((b\circ^2 c)\circ_2 \mathscr{G})\\ &= (a\circ^1 (b\circ^2 c))\circ_1 \mathscr{G}.\end{aligned}$$

by definition. Since $(\mathscr{H},\widetilde{O})$ is associative, we have $(a\circ^1 b)\circ^2 c = a\circ^1 (b\circ^2 c)$. Whence, we get that

$$((a\circ_1 \mathscr{G})\circ^1 (b\circ_2 \mathscr{G}))\circ^2 (c\circ_3 \mathscr{G}) = (a\circ_1 \mathscr{G})\circ^1 ((b\circ_2 \mathscr{G})\circ^2 (c\circ_3 \mathscr{G})),$$

i.e., $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})};\widetilde{O})$ is an associative multi-operation system.

If any element in R has an inverse in $(\mathscr{H};\circ')$, then we know that $\mathscr{G}$ is a unit and $a^{-1}\circ'\mathscr{G}$ is the inverse element of $a\circ'\mathscr{G}$ in the system $(\frac{\mathscr{H}}{\mathscr{G}}|_{(R,\widetilde{P})},\circ')$, namely, it is a group again. □

Let $\mathcal{I}(\widetilde{O})$ be the set of all units $1_\circ, \circ \in \widetilde{O}$ in a multi-operation system $(\mathscr{H};\widetilde{O})$. Define a *multi-kernel* $\widetilde{\mathrm{Ker}}\omega$ of a homomorphism $\omega : (\mathscr{H}_1;\widetilde{O}_1) \to (\mathscr{H}_2;\widetilde{O}_2)$ by

$$\widetilde{\mathrm{Ker}}\omega = \{\ a \in \mathscr{H}_1 \mid \omega(a) = 1_\circ \in \mathcal{I}(\widetilde{O}_2)\ \}.$$

Then we know the homomorphism theorem for multi-operation systems in the following.

Theorem 4.2 *Let ω be an onto homomorphism from associative systems $(\mathscr{H}_1;\widetilde{O}_1)$ to $(\mathscr{H}_2;\widetilde{O}_2)$ with $(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)$ an algebraic system with unit $1_{\circ^-}$ for $\forall\circ^- \in \widetilde{O}_2$ and inverse x^{-1} for $\forall x \in (\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^-)$. Then there are representation pairs $(R_1,\widetilde{P}_1)$ and $(R_2,\widetilde{P}_2)$, where $\widetilde{P}_1 \subset \widetilde{O}, \widetilde{P}_2 \subset \widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}}\omega;\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)} \cong \frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$$

if each element of $\widetilde{\mathrm{Ker}}\omega$ has an inverse in $(\mathscr{H}_1;\circ)$ for $\circ \in \widetilde{O}_1$.

Proof Notice that $\widetilde{\mathrm{Ker}}\omega$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. In fact, for $\forall k_1,k_2 \in \widetilde{\mathrm{Ker}}\omega$ and $\forall\circ \in \widetilde{O}_1$, there is an operation $\circ^- \in \widetilde{O}_2$ such that

$$\omega(k_1\circ k_2) = \omega(k_1)\circ^- \omega(k_2) \in \mathcal{I}(\widetilde{O}_2)$$

since $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system. Whence, $\widetilde{\mathrm{Ker}}\omega$ is an associative subsystem of $(\mathscr{H}_1;\widetilde{O}_1)$. By assumption, for any operation $\circ \in \widetilde{O}_1$ each element $a \in \widetilde{\mathrm{Ker}}\omega$ has an inverse a^{-1} in $(\mathscr{H}_1;\circ)$. Let $\omega : (\mathscr{H}_1;\circ) \to (\mathscr{H}_2;\circ^-)$. We know that

$$\omega(a\circ a^{-1})=\omega(a)\circ^{-}\omega(a^{-1})=1_{\circ^{-}},$$

i.e., $\omega(a^{-1})=\omega(a)^{-1}$ in $(\mathscr{H}_2;\circ^{-})$. Because $\mathcal{I}(\widetilde{O}_2)$ is an algebraic system with an inverse x^{-1} for $\forall x\in\mathcal{I}(\widetilde{O}_2)$ in $((\mathcal{I}(\widetilde{O}_2);\circ^{-})$, we find that $\omega(a^{-1})\in\mathcal{I}(\widetilde{O}_2)$, namely, $a^{-1}\in\widetilde{\mathrm{Ker}\omega}$.

Define a mapping $\sigma:\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}\rightarrow\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$ by

$$\sigma(a\circ\mathrm{Ker}\omega)=\sigma(a)\circ^{-}\mathcal{I}(\widetilde{O}_2)$$

for $\forall a\in R_1,\circ\in\widetilde{P}_1$, where $\omega:(\mathscr{H}_1;\circ)\rightarrow(\mathscr{H}_2;\circ^{-})$. We prove σ is an isomorphism. Notice that σ is onto by that ω is an onto homomorphism. Now if $a\circ_1\widetilde{\mathrm{Ker}\omega}\neq b\circ_2\mathrm{Ker}(\varpi)$ for $a,b\in R_1$ and $\circ_1,\circ_2\in\widetilde{P}_1$, then $\omega(a)\circ_1^{-}\mathcal{I}(\widetilde{O}_2)\neq\omega(b)\circ_2^{-}\mathcal{I}(\widetilde{O}_2)$. Otherwise, we find that $a\circ_1\widetilde{\mathrm{Ker}\omega}=b\circ_2\widetilde{\mathrm{Ker}\omega}$, a contradiction. Whence, $\sigma(a\circ_1\widetilde{\mathrm{Ker}\omega})\neq\sigma(b\circ_2\widetilde{\mathrm{Ker}\omega})$, i.e., σ is a bijection from $\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$.

Since ω is a homomorphism, we get that

$$\begin{aligned}\sigma((a\circ_1\widetilde{\mathrm{Ker}\omega})\circ(b\circ_2\widetilde{\mathrm{Ker}\omega})) &= \sigma(a\circ_1\widetilde{\mathrm{Ker}\omega})\circ^{-}\sigma(b\circ_2\widetilde{\mathrm{Ker}\omega})\\ &= (\omega(a)\circ_1^{-}\mathcal{I}(\widetilde{O}_2))\circ^{-}(\omega(b)\circ_2^{-}\mathcal{I}(\widetilde{O}_2))\\ &= \sigma((a\circ_1\widetilde{\mathrm{Ker}\omega})\circ^{-}\sigma(b\circ_2\widetilde{\mathrm{Ker}\omega}),\end{aligned}$$

i.e., σ is an isomorphism from $\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}$ to $\frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}$. □

Corollary 4.1 *Let $(\mathscr{H}_1;\widetilde{O}_1)$, $(\mathscr{H}_2;\widetilde{O}_2)$ be multi-operation systems with groups $(\mathscr{H}_2;\circ_1)$, $(\mathscr{H}_2;\circ_2)$ for $\forall\circ_1\in\widetilde{O}_1$, $\forall\circ_2\in\widetilde{O}_2$ and $\omega:(\mathscr{H}_1;\widetilde{O}_1)\rightarrow(\mathscr{H}_2;\widetilde{O}_2)$ a homomorphism. Then there are representation pairs $(R_1,\widetilde{P}_1)$ and $(R_2,\widetilde{P}_2)$, where $\widetilde{P}_1\subset\widetilde{O}_1,\widetilde{P}_2\subset\widetilde{O}_2$ such that*

$$\frac{(\mathscr{H}_1;\widetilde{O}_1)}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O}_1)}|_{(R_1,\widetilde{P}_1)}\ \cong\ \frac{(\mathscr{H}_2;\widetilde{O}_2)}{(\mathcal{I}(\widetilde{O}_2);\widetilde{O}_2)}|_{(R_2,\widetilde{P}_2)}.$$

Particularly, if $(\mathscr{H}_2;\widetilde{O}_2)$ is a group, we get an interesting result following.

Corollary 4.2 *Let $(\mathscr{H};\widetilde{O})$ be a multi-operation system and $\omega:(\mathscr{H};\widetilde{O})\rightarrow(\mathscr{A};\circ)$ a onto homomorphism from $(\mathscr{H};\widetilde{O})$ to a group $(\mathscr{A};\circ)$. Then there are representation pairs $(R,\widetilde{P})$, $\widetilde{P}\subset\widetilde{O}$ such that*

$$\frac{(\mathscr{H};\widetilde{O})}{(\widetilde{\mathrm{Ker}\omega};\widetilde{O})}|_{(R,\widetilde{P})}\ \cong\ (\mathscr{A};\circ).$$

§5. Multi-Rings

An associative multi-operation system $(\mathscr{H};\mathcal{O}_1\hookrightarrow\mathcal{O}_2)$ is said to be a *multi-group* if $(\mathscr{H};\circ)$ is a group for $\forall\circ\in\mathcal{O}_1\cup\mathcal{O}_2$, a *multi-ring* (or *multi-field*) if $\mathcal{O}_1=\{\cdot_i|1\leq i\leq l\}$, $\mathcal{O}_2=\{+_i|1\leq i\leq l\}$ with rings (or multi-field) $(\mathscr{H};+_i,\cdot_i)$ for $1\leq i\leq l$. We call them *l-group*, *l-ring* or *l-field*

for abbreviation. It is obvious that a multi-group is a group if $|\mathcal{O}_1 \cup \mathcal{O}_2| = 1$ and a ring or field if $|\mathcal{O}_1| = |\mathcal{O}_2| = 1$ in classical algebra. Likewise, We also denote these units of a l-ring $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ by $1_{\cdot_i}$ and 0_{+_i} in the ring $(\mathscr{H}; +_i, \cdot_i)$. Notice that for $\forall a \in \mathscr{H}$, by these distribute laws we find that

$$a \cdot_i b = a \cdot_i (b +_i 0_{+_i}) = a \cdot_i b +_i a \cdot_i 0_{+_i},$$

$$b \cdot_i a = (b +_i 0_{+_i}) \cdot_i a = b \cdot_i a +_i 0_{+_i} \cdot_i a$$

for $\forall b \in \mathscr{H}$. Whence,

$$a \cdot_i 0_{+_i} = 0_{+_i} \text{ and } 0_{+_i} \cdot_i a = 0_{+_i}.$$

Similarly, a multi-operation subsystem of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is said a *multi-subgroup*, *multi-subring* or *multi-subfield* if it is a *multi-group*, *multi-ring* or *multi-field* itself.

Now let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ be an associative multi-operation system. We find these criterions for multi-subgroups and multi-subrings of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ in the following.

Theorem 5.1 *Let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2$ be a multi-group, $\mathcal{H} \subset \mathscr{H}$. Then $(\mathcal{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is a*

(i) multi-subgroup if and only if for $\forall a, b \in \mathcal{H},\ \circ \in \mathcal{O}_1 \cup \mathcal{O}_2,\ a \circ b_\circ^{-1} \in \mathcal{H}$;

(ii) multi-subring if and only if for $\forall a, b \in \mathcal{H},\ \cdot_i \in \mathcal{O}_1$ and $\forall +_i \in \mathcal{O}_2),\ a \cdot_i b,\ a +_i b_{+_i}^{-1} \in \mathcal{H}$, particularly, a multi-field if $a \cdot_i b_{\cdot_i}^{-1},\ a +_i b_{+_i}^{-1} \in \mathcal{H}$, where, $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$.

Proof The necessity of conditions (i) and (ii) is obvious. Now we consider their sufficiency.

For (i), we only need to prove that $(\mathcal{H}; \circ)$ is a group for $\forall \circ \in \mathcal{O}_1 \cup \mathcal{O}_2$. In fact, it is associative by the definition of multi-groups. For $\forall a \in \mathcal{H}$, we get that $1_\circ = a \circ a_\circ^{-1} \in \mathcal{H}$ and $1_\circ \circ a_\circ^{-1} \in \mathcal{H}$. Whence, $(\mathcal{H}; \circ)$ is a group.

Similarly for (ii), the conditions $a \cdot_i b,\ a +_i b_{+_i}^{-1} \in \mathcal{H}$ imply that $(\mathcal{H}; +_i)$ is a group and closed in operation $\cdot_i \in \mathcal{O}_1$. These associative or distributive laws are hold by $(\mathscr{H}; +_i, \cdot_i)$ being a ring for any integer $i, 1 \leq i \leq l$. Particularly, $a \cdot_i b_{\cdot_i}^{-1} \in \mathcal{H}$ imply that $(\mathcal{H}; \cdot_i)$ is also a group. Whence, $(\mathscr{H}; +_i, \cdot_i)$ is a field for any integer $i, 1 \leq i \leq l$ in this case. □

A multi-ring $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ with $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$ is *integral* if for $\forall a, b \in \mathscr{H}$ and an integer $i,\ 1 \leq i \leq l$, $a \circ_i b = b \circ_i a$, $1_{\circ_i} \neq 0_{+i}$ and $a \circ_i b = 0_{+_i}$ implies that $a = 0_{+_i}$ or $b = 0_{+_i}$. If $l = 1$, an integral l-ring is the integral ring by definition. For the case of multi-rings with finite elements, an integral multi-ring is nothing but a multi-field. See the next result.

Theorem 5.2 *A finitely integral multi-ring is a multi-field.*

Proof Let $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ be a finitely integral multi-ring with $\mathscr{H} = \{a_1, a_2 \cdots, a_n\}$, where $\mathcal{O}_1 = \{\cdot_i | 1 \leq i \leq l\}$, $\mathcal{O}_2 = \{+_i | 1 \leq i \leq l\}$. For any integer $i, 1 \leq i \leq l$, choose an element $a \in \mathscr{H}$ and $a \neq 0_{+_i}$. Then

$$a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n$$

are n elements. If $a \circ_i a_s = a \circ_i a_t$, i.e., $a \circ_i (a_s +_i a_t^{-1}) = 0_{+_i}$. By definition, we know that

$a_s +_i a_t^{-1} = 0+_i$, namely, $a_s = a_t$. That is, these $a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n$ are different two by two. Whence,

$$\mathscr{H} = \{\ a \circ_i a_1,\ a \circ_i a_2, \cdots,\ a \circ_i a_n\ \}.$$

Now assume $a \circ_i a_s = 1_{\cdot_i}$, then $a^{-1} = a_s$, i.e., each element of $\mathscr{H}$ has an inverse in $(\mathscr{H}; \cdot_i)$, which implies it is a commutative group. Therefore, $(\mathscr{H}; +_i, \cdot_i)$ is a field for any integer $i, 1 \leq i \leq l$. □

Corollary 5.1 *Any finitely integral ring is a field.*

Let $(\mathscr{H}; \mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)$, $(\mathscr{H}; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ be multi-rings with $\mathcal{O}_1^k = \{\cdot_i^k | 1 \leq i \leq l_k\}$, $\mathcal{O}_2^k = \{+_i^k | 1 \leq i \leq l_k\}$ for $k = 1, 2$ and $\varrho : (\mathscr{H}; \mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1) \rightarrow (\mathscr{H}; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ a homomorphism. Define a *zero kernel* $\widetilde{\mathrm{Ker}}\varrho$ of ϱ by

$$\widetilde{\mathrm{Ker}_0}\varrho = \{a \in \mathscr{H} | \varrho(a) = 0_{+_i^2}, 1 \leq i \leq l_2\}.$$

Then, for $\forall h \in \mathscr{H}$ and $a \in \widetilde{\mathrm{Ker}_0}\varrho$, $\varrho(a \cdot_i^1 h) = 0_{+_i} \varrho(\cdot_i)h = 0_{+_i}$, i.e., $a \cdot_i h \in \widetilde{\mathrm{Ker}_0}\varrho$. Similarly, $h \cdot_i a \in \widetilde{\mathrm{Ker}_0}\varrho$. These properties imply the conception of multi-ideals of a multi-ring introduced following.

Choose a subset $\mathcal{I} \subset \mathscr{H}$. For $\forall h \in \mathscr{H},\ a \in \mathcal{I}$, if there are

$$h \circ_i a \in \mathcal{I} \ \text{ and } \ a \circ_i h \in \mathcal{H},$$

then $\mathcal{I}$ is said a *multi-ideal*. Previous discussion shows that the zero kernel $\widetilde{\mathrm{Ker}_0}\varrho$ of a homomorphism ϱ on a multi-ring is a multi-ideal. Now let $\mathcal{I}$ be a multi-ideal of $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$. According to Corollary 4.1, we know that there is a representation pair (R_2, P_2) such that

$$\widetilde{\mathcal{I}} = \{a +_i \mathcal{I} \mid a \in R_2,\ +_i \in P_2\}$$

is a commutative multi-group. By the distributive laws, we find that

$$\begin{aligned}(a +_i \mathcal{I}) \cdot_j (b +_k \mathcal{I}) &= a \cdot_j b +_k a \cdot_j \mathcal{I} +_i \mathcal{I} b +_k \mathcal{I} \cdot_j \mathcal{I} \\ &= a \cdot_j b +_k \mathcal{I}.\end{aligned}$$

Similarly, we also know these associative and distributive laws follow in $(\widetilde{\mathcal{I}}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$. Whence, $(\widetilde{\mathcal{I}}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is also a multi-ring, called the *quotient multi-ring of* $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$, denoted by $(\mathscr{H} : \mathcal{I})$.

Define a mapping $\varrho : (\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2) \rightarrow (\mathscr{H} : \mathcal{I})$ by $\varrho(a) = a +_i \mathcal{I}$ for $\forall a \in \mathscr{H}$ if $a \in a +_i \mathcal{I}$. Then it can be checked immediately that it is a homomorphism with

$$\widetilde{\mathrm{Ker}_0}\varrho \ = \ \mathcal{I}.$$

Therefore, we conclude that *any multi-ideal is a zero kernel of a homomorphism on a multi-ring*. The following result is a special case of Theorem 4.2.

Theorem 5.3 *Let* $(\mathscr{H}_1; \mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)$ *and* $(\mathscr{H}_2; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ *be multi-rings and* $\omega : (\mathscr{H}_1; \mathcal{O}_2^1) \to (\mathscr{H}_2; \mathcal{O}_2^2)$ *be an onto homomorphism with* $(\mathcal{I}(\mathcal{O}_2^2); \mathcal{O}_2^2)$ *be a multi-operation system, where* $\mathcal{I}(\mathcal{O}_2^2)$ *denotes all units in* $(\mathscr{H}_2; \mathcal{O}_2^2)$. *Then there exist representation pairs* $(R_1, \widetilde{P}_1), (R_2, \widetilde{P}_2)$ *such that*

$$(\mathscr{H} : \mathcal{I})|_{(R_1, \widetilde{P}_1)} \cong \frac{(\mathscr{H}_2; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)}{(\mathcal{I}(\mathcal{O}_2^2); \mathcal{O}_2^2)}|_{(R_2, \widetilde{P}_2)}.$$

Particularly, if $(\mathscr{H}_2; \mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)$ is a ring, we get an interesting result following.

Corollary 5.2 *Let* $(\mathscr{H}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ *be a multi-ring,* $(R; +, \cdot)$ *a ring and* $\omega : (\mathscr{H}; \mathcal{O}_2) \to (R; +)$ *be an onto homomorphism. Then there exists a representation pair* $(R, \widetilde{P})$ *such that*

$$(\mathscr{H} : \mathcal{I})|_{(R, \widetilde{P})} \cong (R; +, \cdot).$$

§6. Finite Dimensional Multi-Modules

Let $\mathcal{O} = \{ +_i \mid 1 \le i \le m\}$, $\mathcal{O}_1 = \{\cdot_i | 1 \le i \le m\}$ and $\mathcal{O}_2 = \{\dot{+}_i | 1 \le i \le m\}$ be operation sets, $(\mathscr{M}; \mathcal{O})$ a commutative m-group with units 0_{+_i} and $(\mathscr{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ a multi-ring with a unit 1. for $\forall \cdot \in \mathcal{O}_1$. For any integer i, $1 \le i \le m$, define a binary operation $\times_i : \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ by $a \times_i x$ for $a \in \mathscr{R}$, $x \in \mathscr{M}$ such that for $\forall a, b \in \mathscr{R}$, $\forall x, y \in \mathscr{M}$, conditions following hold:

(*i*) $a \times_i (x +_i y) = a \times_i x +_i a \times_i y$;

(*ii*) $(a \dot{+}_i b) \times_i x = a \times_i x +_i b \times_i x$;

(*iii*) $(a \cdot_i b) \times_i x = a \times_i (b \times_i x)$;

(*iv*) $1_{\cdot_i} \times_i x = x$.

Then $(\mathscr{M}; \mathcal{O})$ is said an *algebraic multi-module over* $(\mathscr{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ abbreviated to an m-*module* and denoted by $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$. In the case of $m = 1$, It is obvious that $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ is a *module*, particularly, if $(\mathscr{R}; \mathcal{O}_1 \hookrightarrow \mathcal{O}_2)$ is a field, then $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ is a *linear space* in classical algebra.

For any integer k, $a_i \in \mathscr{R}$ and $x_i \in \mathscr{M}$, where $1 \le i,\ k \le s$, equalities following are hold by induction on the definition of m-modules.

$$a \times_k (x_1 +_k x_2 +_k \cdots +_k x_s) = a \times_k x_1 +_k a \times_k x_2 +_k \cdots +_k a_s \times_k x,$$

$$(a_1 \dot{+}_k a_2 \dot{+}_k \cdots \dot{+}_k a_s) \times_k x = a_1 \times_k x +_k a_2 \times_k x +_k \cdots +_k a_s \times_k x,$$

$$(a_1 \cdot_k a_2 \cdot_k \cdots \cdot_k a_s) \times_k x = a_1 \times_k (a_2 \times_k \cdots \times (a_s \times_k x) \cdots)$$

and

$$1_{\cdot_{i_1}} \times_{i_1} (1_{\cdot_{i_2}} \times_{i_2} \cdots \times_{i_{s-1}} (1_{\cdot_{i_s}} \times_{i_s} x) \cdots) = x$$

for integers $i_1, i_2, \cdots, i_s \in \{1, 2, \cdots, m\}$.

Notice that for $\forall a, x \in \mathscr{M}$, $1 \le i \le m$,

$$a \times_i x = a \times_i (x +_i 0_{+_i}) = a \times_i x +_i a \times_i 0_{+_i},$$

we find that $a \times_i 0_{+_i} = 0_{+_i}$. Similarly, $0_{\dot{+}_i} \times_i a = 0_{+_i}$. Applying this fact, we know that

$$a \times_i x +_i a^-_{\dot{+}_i} \times_i x = (a \dot{+}_i a^-_{\dot{+}_i}) \times_i x = 0_{\dot{+}_i} \times_i x = 0_{+_i}$$

and

$$a \times_i x +_i a \times_i x^-_{+_i} = a \times_i (x +_i x^-_{+_i}) = a \times_i 0_{+_i} = 0_{+_i}.$$

We know that

$$(a \times_i x)^-_{+_i} = a^-_{\dot{+}_i} \times_i x = a \times_i x^-_{+_i}.$$

Notice that $a \times_i x = 0_{+_i}$ does not always mean $a = 0_{\dot{+}_i}$ or $x = 0_{+_i}$ in an m-module $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ unless $a^-_{\dot{+}_i}$ is existing in $(\mathscr{R}; \dot{+}_i, \cdot_i)$ if $x \neq 0_{+_i}$.

Now choose $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1))$ an m-module with operation sets $\mathcal{O}_1 = \{ +'_i \mid 1 \leq i \leq m\}$, $\mathcal{O}_1^1 = \{\cdot^1_i | 1 \leq i \leq m\}$, $\mathcal{O}_2^1 = \{\dot{+}^1_i | 1 \leq i \leq m\}$ and $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ an n-module with operation sets $\mathcal{O}_2 = \{ +''_i \mid 1 \leq i \leq n\}$, $\mathcal{O}_1^2 = \{\cdot^2_i | 1 \leq i \leq n\}$, $\mathcal{O}_2^2 = \{\dot{+}^2_i | 1 \leq i \leq n\}$. They are said *homomorphic* if there is a mapping $\iota : \mathscr{M}_1 \to \mathscr{M}_2$ such that for any integer $i, 1 \leq i \leq m$,

(i) $\iota(x +'_i y) = \iota(x) +'' \iota(y)$ for $\forall x, y \in \mathscr{M}_1$, where $\iota(+'_i) = +'' \in \mathcal{O}_2$;

(ii) $\iota(a \times_i x) = a \times_i \iota(x)$ for $\forall x \in \mathscr{M}_1$.

If ι is a bijection, these modules $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1))$ and $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2))$ are said to be *isomorphic*, denoted by

$$\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1) : \mathscr{R}_1(\mathcal{O}_1^1 \hookrightarrow \mathcal{O}_2^1)) \cong \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2) : \mathscr{R}_2(\mathcal{O}_1^2 \hookrightarrow \mathcal{O}_2^2)).$$

Let $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ be an m-module. For a multi-subgroup $(\mathscr{N}; \mathcal{O})$ of $(\mathscr{M}; \mathcal{O})$, if for any integer i, $1 \leq i \leq m$, $a \times_i x \in \mathscr{N}$ for $\forall a \in \mathscr{R}$ and $x \in \mathscr{N}$, then by definition it is itself an m-module, called a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$.

Now if $\mathbf{Mod}(\mathscr{N}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$, by Theorem 4.2, we can get a quotient multi-group $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ with a representation pair $(R, \widetilde{P})$ under operations

$$(a +_i \mathscr{N}) + (b +_j \mathscr{N}) = (a + b) +_i \mathscr{N}$$

for $\forall a, b \in R, + \in \mathcal{O}$. For convenience, we denote elements $x +_i \mathscr{N}$ in $\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}$ by $\overline{x^{(i)}}$. For an integer $i, 1 \leq i \leq m$ and $\forall a \in \mathscr{R}$, define

$$a \times_i \overline{x^{(i)}} = \overline{(a \times_i x)^{(i)}}.$$

Then it can be shown immediately that

(i) $a \times_i (\overline{x^{(i)}} +_i \overline{y^{(i)}}) = a \times_i \overline{x^{(i)}} +_i a \times_i \overline{y^{(i)}}$;

(ii) $(a \dot{+}_i b) \times_i \overline{x^{(i)}} = a \times_i \overline{x^{(i)}} +_i b \times_i \overline{x^{(i)}}$;

(iii) $(a \cdot_i b) \times_i \overline{x^{(i)}} = a \times_i (b \times_i \overline{x^{(i)}})$;

(iv) $1_{\cdot_i} \times_i \overline{x^{(i)}} = \overline{x^{(i)}}$,

i.e.,$(\frac{\mathscr{M}}{\mathscr{N}}|_{(R,\widetilde{P})}:\mathscr{R})$ is also an m-module, called a quotient module of $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ to $\mathbf{Mod}(\mathscr{N}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$. Denoted by $\mathbf{Mod}(\mathscr{M}/\mathscr{N})$.

The result on homomorphisms of m-modules following is an immediately consequence of Theorem 4.2.

Theorem 6.1 *Let $\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))$, $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ be multi-modules with $\mathcal{O}_1=\{\ +_i'\ |\ 1\le i\le m\}$, $\mathcal{O}_2=\{\ +_i''\ |\ 1\le i\le n\}$, $\mathcal{O}_1^1=\{\cdot_i^1|1\le i\le m\}$, $\mathcal{O}_2^1=\{\dot{+}_i^1|1\le i\le m\}$, $\mathcal{O}_1^2=\{\cdot_i^2|1\le i\le n\}$, $\mathcal{O}_2^2=\{\dot{+}_i^2|1\le i\le n\}$ and $\iota:\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))\to\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ be a onto homomorphism with $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ a multi-group, where $\mathcal{I}(\mathcal{O}_2^2)$ denotes all units in the commutative multi-group $(\mathscr{M}_2;\mathcal{O}_2)$. Then there exist representation pairs $(R_1,\widetilde{P}_1),(R_2,\widetilde{P}_2)$ such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))|_{(R_2,\widetilde{P}_2)},$$

where $\mathscr{N}=\mathrm{Ker}\iota$ is the kernel of ι. Particularly, if $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ is trivial, i.e., $|\mathcal{I}(\mathcal{O}_2)|=1$, then

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))|_{(R_2,\widetilde{P}_2)}.$$

Proof Notice that $(\mathcal{I}(\mathcal{O}_2);\mathcal{O}_2)$ is a commutative multi-group. We can certainly construct a quotient module $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))$. Applying Theorem 2.3.6, we find that

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R_1,\widetilde{P}_1)}\ \cong\ \mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))|_{(R_2,\widetilde{P}_2)}.$$

Notice that $\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2)/\mathcal{I}(\mathcal{O}_2))=\mathbf{Mod}(\mathscr{M}_2(\mathcal{O}_2):\mathscr{R}_2(\mathcal{O}_1^2\hookrightarrow\mathcal{O}_2^2))$ in the case of $|\mathcal{I}(\mathcal{O}_2)|=1$. We get the isomorphism as desired. □

Corollary 6.1 *Let $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ be an m-module with $\mathcal{O}=\{\ +_i\ |\ 1\le i\le m\}$, $\mathcal{O}_1=\{\cdot_i|1\le i\le m\}$, $\mathcal{O}_2=\{\dot{+}_i|1\le i\le m\}$, M a module on a ring $(R;+,\cdot)$ and $\iota:\mathbf{Mod}(\mathscr{M}_1(\mathcal{O}_1):\mathscr{R}_1(\mathcal{O}_1^1\hookrightarrow\mathcal{O}_2^1))\to M$ a onto homomorphism with $\mathrm{Ker}\iota=\mathscr{N}$. Then there exists a representation pair $(R',\widetilde{P})$ such that*

$$\mathbf{Mod}(\mathscr{M}/\mathscr{N})|_{(R',\widetilde{P})}\ \cong\ M,$$

particularly, if $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ is a module $\mathscr{M}$, then

$$\mathscr{M}/\mathscr{N}\ \cong\ M.$$

For constructing multi-submodules of an m-module $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ with $\mathcal{O}=\{\ +_i\ |\ 1\le i\le m\}$, $\mathcal{O}_1=\{\cdot_i|1\le i\le m\}$, $\mathcal{O}_2=\{\dot{+}_i|1\le i\le m\}$, a general way is described in the following.

Let $\widehat{S}\subset\mathscr{M}$ with $|\widehat{S}|=n$. Define its *linearly spanning set* $\left\langle\widehat{S}|\mathscr{R}\right\rangle$ in $\mathbf{Mod}(\mathscr{M}(\mathcal{O}):\mathscr{R}(\mathcal{O}_1\hookrightarrow\mathcal{O}_2))$ to be

$$\left\langle\widehat{S}|\mathscr{R}\right\rangle=\{\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\alpha_{ij}\times_i x_{ij}|\ \alpha_{ij}\in\mathscr{R},x_{ij}\in\widehat{S}\ \},$$

where

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} a_{ij}\times_{ij} x_i = a_{11}\times_1 x_{11} +_1 \cdots +_1 a_{1n}\times_1 x_{1n}$$
$$+^{(1)} a_{21}\times_2 x_{21} +_2 \cdots +_2 a_{2n}\times_2 x_{2n}$$
$$+^{(2)} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +^{(3)}$$
$$a_{m1}\times_m x_{m1} +_m \cdots +_m a_{mn}\times_m x_{mn}$$

with $+^{(1)}, +^{(2)}, +^{(3)} \in \mathscr{O}$ and particularly, if $+_1 = +_2 = \cdots = +_m$, it is denoted by $\sum\limits_{i=1}^{m} x_i$ as usual. It can be checked easily that $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is a multi-submodule of $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, call it *generated by* $\widehat{S}$ *in* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$. If $\widehat{S}$ is finite, we also say that $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is *finitely generated*. Particularly, if $\widehat{S} = \{x\}$, then $\left\langle \widehat{S}|\mathscr{R}\right\rangle$ is called a *cyclic multi-submodule of* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, denoted by $\mathscr{R}x$. Notice that

$$\mathscr{R}x = \{ \bigoplus_{i=1}^{m} a_i \times_i x \mid a_i \in \mathscr{R} \}$$

by definition. For any finite set $\widehat{S}$, if for any integer $s, 1 \le s \le m$,

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{s_i} \alpha_{ij}\times_i x_{ij} = 0_{+_s}$$

implies that $\alpha_{ij} = 0_{\dot{+}_s}$ for $1 \le i \le m, 1 \le j \le n$, then we say that $\{x_{ij}|1 \le i \le m, 1 \le j \le n\}$ is independent and $\widehat{S}$ a *basis of the multi-module* $\mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$, denoted by $\left\langle \widehat{S}|\mathscr{R}\right\rangle = \mathbf{Mod}(\mathscr{M}(\mathscr{O}) : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$.

For a multi-ring $(\mathscr{R}; \mathscr{O}_1 \hookrightarrow \mathscr{O}_2)$ with a unit 1. for $\forall \cdot \in \mathscr{O}_1$, where $\mathscr{O}_1 = \{\cdot_i|1 \le i \le m\}$ and $\mathscr{O}_2 = \{\dot{+}_i|1 \le i \le m\}$, let

$$\mathscr{R}^{(n)} = \{(x_1, x_2, \cdots, x_n)|\ x_i \in \mathscr{R}, 1 \le i \le n\}.$$

Define operations

$$(x_1, x_2, \cdots, x_n) +_i (y_1, y_2, \cdots, y_n) = (x_1 \dot{+}_i y_1, x_2 \dot{+}_i y_2, \cdots, x_n \dot{+}_i y_n)$$

and

$$a \times_i (x_1, x_2, \cdots, x_n) = (a \cdot_i x_1, a \cdot_i x_2, \cdots, a \cdot_i x_n)$$

for $\forall a \in \mathscr{R}$ and integers $1 \le i \le m$. Then it can be immediately known that $\mathscr{R}^{(n)}$ is a multi-module $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathscr{O}_1 \hookrightarrow \mathscr{O}_2))$. We construct a basis of this special multi-module in the following.

For any integer $k, 1 \le k \le n$, let

$$\mathbf{e}_1 = (1_{\cdot_k}, 0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k});$$

$$\mathbf{e}_2 = (0_{\dot{+}_k}, 1_{\cdot_k}, \cdots, 0_{\dot{+}_k});$$

$$\cdots\cdots\cdots\cdots\cdots\cdots ;$$

$$\mathbf{e}_n = (0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k}, 1_{\cdot_k}).$$

Notice that

$$(x_1, x_2, \cdots, x_n) = x_1 \times_k \mathbf{e}_1 +_k x_2 \times_k \mathbf{e}_2 +_k \cdots +_k x_n \times_k \mathbf{e}_n.$$

We find that each element in $\mathscr{R}^{(n)}$ is generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$. Now since

$$(x_1, x_2, \cdots, x_n) = (0_{\dot{+}_k}, 0_{\dot{+}_k}, \cdots, 0_{\dot{+}_k})$$

implies that $x_i = 0_{\dot{+}_k}$ for any integer $i, 1 \leq i \leq n$. Whence, $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is a basis of $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$.

Theorem 6.2 *Let* $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) = \left\langle \widehat{S} | \mathscr{R} \right\rangle$ *be a finitely generated multi-module with* $\widehat{S} = \{u_1, u_2, \cdots, u_n\}$*. Then*

$$\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)) \ \cong\ \mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2)).$$

Proof Define a mapping $\vartheta : \mathscr{M}(\mathcal{O}) \to \mathscr{R}^{(n)}$ by $\vartheta(u_i) = \mathbf{e}_i$, $\vartheta(a \times_j u_i) = a \times_j \mathbf{e}_j$ and $\vartheta(u_i +_k u_j) = \mathbf{e}_i +_k \mathbf{e}_j$ for any integers i, j, k, where $1 \leq i, j, k \leq n$. Then we know that

$$\vartheta(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} \times_i u_i) = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} a_{ij} \times_i \mathbf{e}_i.$$

Whence, ϑ is a homomorphism. Notice that it is also $1-1$ and onto. We know that ϑ is an isomorphism between $\mathbf{Mod}(\mathscr{M}(\mathcal{O}) : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$ and $\mathbf{Mod}(\mathscr{R}^{(n)} : \mathscr{R}(\mathcal{O}_1 \hookrightarrow \mathcal{O}_2))$. □

§7. Combinatorially Algebraic Systems

An *algebraic multi-system* is a pair $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ with

$$\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i \ \text{ and } \ \widetilde{\mathscr{O}} = \bigcup_{i=1}^{m} \mathcal{O}_i$$

such that for any integer $i, 1 \leq i \leq m$, $(\mathscr{H}_i; \mathcal{O}_i)$ is a multi-operation system. For an algebraic multi-operation system $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ and an integer $i, 1 \leq i \leq m$, a homomorphism $p_i : (\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}}) \to (\mathscr{H}_i; \mathcal{O}_i)$ is called a *sectional projection*, which is useful in multi-systems.

Two multi-systems $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}_1})$, $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}_2})$, where $\widetilde{\mathscr{A}_1} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$ and $\widetilde{\mathscr{O}_1} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ for $k = 1,\ 2$ are *homomorphic* if there is a mapping $o : \widetilde{\mathscr{A}_1} \to \widetilde{\mathscr{A}_2}$ such that op_i is a homomorphism for any integer $i, 1 \leq i \leq m$. By this definition, we know the existent conditions for homomorphisms on algebraic multi-systems following.

Theorem 7.1 *There exists a homomorphism from an algebraic multi-system* $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}_1})$ *to* $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}_2})$*, where* $\widetilde{\mathscr{A}_k} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$ *and* $\widetilde{\mathscr{O}_k} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ *for* $k = 1,\ 2$ *if and only if there are homomorphisms* $\eta_1, \eta_2, \cdots, \eta_m$ *on* $(\mathscr{H}_1^1; \mathcal{O}_1^1)$, $(\mathscr{H}_2^1; \mathcal{O}_2^1), \cdots$, $(\mathscr{H}_m^1; \mathcal{O}_m^1)$ *such that*

$$\eta_i|_{\mathscr{H}_i^1\cap\mathscr{H}_j^1} = \eta_j|_{\mathscr{H}_i^1\cap\mathscr{H}_j^1}$$

for any integer $1 \leq i, j \leq m$.

Proof By definition, if there is a homomorphism $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$, then op_i is a homomorphism on $(\mathscr{H}_i^1; \mathcal{O}_i^1)$ for any integer $i, 1 \leq i \leq m$.

On the other hand, if there are homomorphisms $\eta_1, \eta_2, \cdots, \eta_m$ on $(\mathscr{H}_1^1; \mathcal{O}_1^1)$, $(\mathscr{H}_2^1; \mathcal{O}_2^1), \cdots$, $(\mathscr{H}_m^1; \mathcal{O}_m^1)$, define a mapping $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ by $o(a) = \eta_i(a)$ if $a \in \mathscr{H}_i^1$. Then it can be checked immediately that o is a homomorphism. □

Let $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ be a homomorphism with a unit $1_\circ$ for each operation $\circ \in \widetilde{\mathscr{O}}_2$. Similar to the case of multi-operation systems, we define the *multi-kernel* $\widetilde{\mathrm{Ker}}(o)$ by

$$\widetilde{\mathrm{Ker}}(o) = \{\ a \in \widetilde{\mathscr{A}_1} \mid o(a) = 1_\circ \ \text{ for } \ \forall \circ \in \widetilde{\mathscr{O}}_2\ \}.$$

Then we have the homomorphism theorem on algebraic multi-systems following.

Theorem 7.2 *Let* $(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ *be algebraic multi-systems, where* $\widetilde{\mathscr{A}_k} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$, $\widetilde{\mathscr{O}_k} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ *for* $k = 1, 2$ *and* $o : (\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)$ *a onto homomorphism with a multi-group* $(\mathcal{I}_i^2; \mathcal{O}_i^2)$ *for any integer* $i, 1 \leq i \leq m$. *Then there are representation pairs* $(\widetilde{R}_1, \widetilde{P}_1)$ *and* $(\widetilde{R}_2, \widetilde{P}_2)$ *such that*

$$\frac{(\widetilde{\mathscr{A}_1}; \widetilde{\mathscr{O}}_1)}{(\widetilde{\mathrm{Ker}}(o); \mathcal{O}_1)}|_{(\widetilde{R}_1, \widetilde{P}_1)} \cong \frac{(\widetilde{\mathscr{A}_2}; \widetilde{\mathscr{O}}_2)}{(\widetilde{\mathcal{I}}(\mathcal{O}_2); \mathcal{O}_2)}|_{(\widetilde{R}_2, \widetilde{P}_2)}$$

where $(\widetilde{\mathcal{I}}(\mathcal{O}_2); \mathcal{O}_2) = \bigcup\limits_{i=1}^{m} (\mathcal{I}_i^2; \mathcal{O}_i^2)$.

Proof By definition, we know that $o|_{\mathscr{H}_i^1} : (\mathscr{H}_i^1; \mathcal{O}_i^1) \to (\mathscr{H}_{o(i)}^2; \mathcal{O}_{o(i)}^2)$ is also an onto homomorphism for any integer $i, 1 \leq i \leq m$. Applying Theorem 4.2 and Corollary 4.1, we can find representation pairs $(R_i^1, \widetilde{P}_i^1)$ and $(R_i^2, \widetilde{P}_i^2)$ such that

$$\frac{(\mathscr{H}_i^1; \mathcal{O}_i^1)}{(\mathrm{Ker}(o|_{\mathscr{H}_i^1}); \mathcal{O}_i^1)}_{(R_i^1, \widetilde{P}_i^1)} \cong \frac{(\mathscr{H}_{o(i)}^2; \mathcal{O}_{o(i)}^2)}{(\mathcal{I}_{o(i)}^2; \mathcal{O}_{o(i)}^2)}_{(R_{o(i)}^1, \widetilde{P}_{o(i)}^1)}.$$

Notice that

$$\widetilde{\mathscr{A}_k} = \bigcup_{i=1}^{m} \mathscr{H}_i^k, \quad \widetilde{\mathscr{O}_k} = \bigcup_{i=1}^{m} \mathcal{O}_i^k$$

for $k = 1, 2$ and

$$\widetilde{\mathrm{Ker}}(o) = \bigcup_{i=1}^{m} \mathrm{Ker}(o|_{\mathscr{H}_i^1}).$$

We finally get that

$$\frac{(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)}{(\widetilde{\mathrm{Ker}(o)};\mathcal{O}_1)}|_{(\widetilde{R}_1,\widetilde{P}_1)} \cong \frac{(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)}{(\widetilde{\mathcal{I}}(\mathcal{O}_2);\mathcal{O}_2)}|_{(\widetilde{R}_2,\widetilde{P}_2)}$$

with

$$\widetilde{R}_k = \bigcup_{i=1}^{m} R_i^k \ \text{ and } \ \widetilde{P}_k = \bigcup_{i=1}^{m} \widetilde{P}_i^k$$

for $k = 1$ or 2. $\square$

Let $(A;\circ)$ be an algebraic system with operation $\circ$. We associate a *labeled graph* $G^L[A]$ with $(A;\circ)$ by

$$V(G^L[A]) = A,$$
$$E(G^L[A]) = \{(a,c) \text{ with label } \circ b \mid \text{if } a \circ b = c \text{ for } \forall a,b,c \in A\},$$

as shown in Fig.7.1.

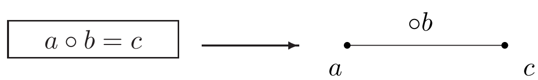


Fig.7.1

The advantage of this diagram on systems is that we can find a$\circ b = c$ for any edge in $G^L[A]$, if its vertices are a,c with a label $\circ b$ and vice versa immediately. For example, the labeled graph $G^L[Z_4]$ of an *Abelian* group Z_4 is shown in Fig.7.2.

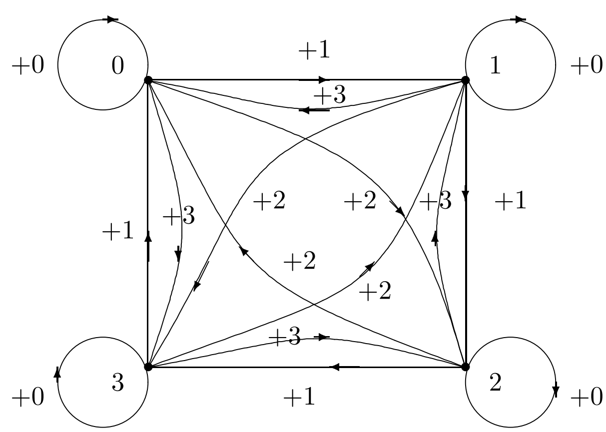


Fig.7.2

Some structure properties on these diagrams $G^L[A]$ of systems are shown in the following.

Property 1. *The labeled graph $G^L[A]$ is connected if and only if there are no partition $A = A_1 \bigcup A_2$ such that for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are no definition for $a_1 \circ a_2$ in $(A;\circ)$.*

If $G^L[A]$ is disconnected, we choose one component C and let $A_1 = V(C)$. Define $A_2 = V(G^L[A]) \setminus V(C)$. Then we get a partition $A = A_1 \bigcup A_2$ and for $\forall a_1 \in A_1$, $\forall a_2 \in A_2$, there are

no definition for $a_1 \circ a_2$ in $(A;\circ)$, a contradiction.

Property 2. *If there is a unit 1_A in $(A;\circ)$, then there exists a vertex 1_A in $G^L[A]$ such that the label on the edge $(1_A, x)$ is $\circ x$.*

For a multiple 2-edge (a,b) in a directed graph, if two orientations on edges are both to a or both to b, then we say it a *parallel multiple* 2*-edge*. If one orientation is to a and another is to b, then we say it an *opposite multiple* 2*-edge*.

Property 3. *For $\forall a \in A$, if $a_\circ^{-1}$ exists, then there is an opposite multiple* 2*-edge $(\mathbf{1}_A, a)$ in $G^L[A]$ with labels $\circ a$ and $\circ a_\circ^{-1}$, respectively.*

Property 4. *For $\forall a, b \in A$ if $a \circ b = b \circ a$, then there are edges (a,x) and (b,x), $x \in A$ in $(A;\circ)$ with labels $w(a,x) = \circ b$ and $w(b,x) = \circ a$ in $G^L[A]$, respectively.*

Property 5. *If the cancellation law holds in $(A;\circ)$, i.e., for $\forall a, b, c \in A$, if $a \circ b = a \circ c$ then $b = c$, then there are no parallel multiple* 2*-edges in $G^L[A]$.*

These properties $2-5$ are gotten by definition. Each of these cases is shown in (1), (2), (3) and (4) in Fig.7.3.

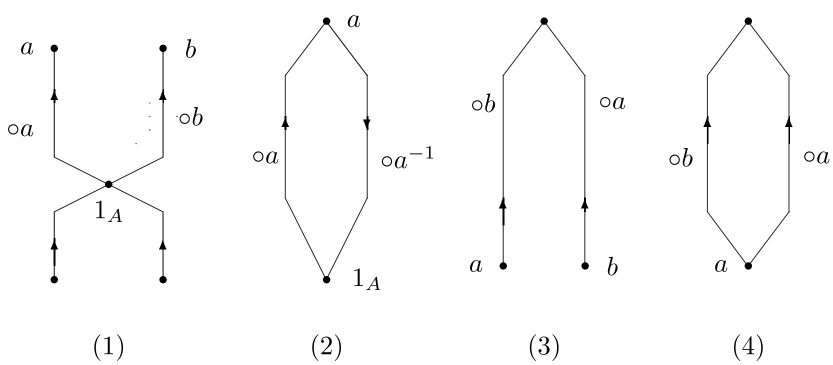


Fig.7.3

Now we consider the diagrams of algebraic multi-systems. Let $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ be an algebraic multi-system with

$$\widetilde{\mathscr{A}} = \bigcup_{i=1}^{m} \mathscr{H}_i \quad \text{and} \quad \widetilde{\mathscr{O}} = \bigcup_{i=1}^{m} \mathcal{O}_i$$

such that $(\mathscr{H}_i;\mathcal{O}_i)$ is a multi-operation system for any integer $i, 1 \le i \le m$, where the operation set $\mathcal{O}_i = \{\circ_{ij} | 1 \le j \le n_i\}$. Define a labeled graph $G^L[\widetilde{\mathscr{A}}]$ associated with $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ by

$$G^L[\widetilde{\mathscr{A}}] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_i} G^L[(\mathscr{H}_i;\circ_{ij})],$$

where $G^L[(\mathscr{H}_i;\circ_{ij})]$ is the associated labeled graph of $(\mathscr{H}_i;\circ_{ij})$ for $1 \le i \le m$, $1 \le j \le n_{ij}$. The importance of $G^L[\widetilde{\mathscr{A}}]$ is displayed in the next result.

Theorem 7.3 *Let $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ be two algebraic multi-systems. Then*

$$(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1) \cong (\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$$

if and only if

$$G^L[\widetilde{\mathscr{A}_1}] \cong G^L[\widetilde{\mathscr{A}_2}].$$

Proof If $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1) \cong (\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$, by definition, there is a $1-1$ mapping $\omega : \widetilde{\mathscr{A}_1} \to \widetilde{\mathscr{A}_2}$ with $\omega : \widetilde{\mathscr{O}}_1 \to \widetilde{\mathscr{O}}_2$ such that for $\forall a, b \in \widetilde{\mathscr{A}_1}$ and $\circ_1 \in \widetilde{\mathscr{O}}_1$, there exists an operation $\circ_2 \in \widetilde{\mathscr{O}}_2$ with the equality following hold,

$$\omega(a \circ_1 b) = \omega(a) \circ_2 \omega(b).$$

Not loss of generality, assume $a \circ_1 b = c$ in $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$. Then for an edge (a, c) with a label $\circ_1 b$ in $G^L[\widetilde{\mathscr{A}_1}]$, there is an edge $(\omega(a), \omega(c))$ with a label $\circ_2\omega(b)$ in $G^L[\widetilde{\mathscr{A}_2}]$, i.e., ω is an equivalence from $G^L[\widetilde{\mathscr{A}_1}]$ to $G^L[\widetilde{\mathscr{A}_2}]$. Therefore, $G^L[\widetilde{\mathscr{A}_1}] \cong G^L[\widetilde{\mathscr{A}_2}]$.

Conversely, if $G^L[\widetilde{\mathscr{A}_1}] \cong G^L[\widetilde{\mathscr{A}_2}]$, let ϖ be a such equivalence from $G^L[\widetilde{\mathscr{A}_1}]$ to $G^L[\widetilde{\mathscr{A}_2}]$, then for an edge (a, c) with a label $\circ_1 b$ in $G^L[\widetilde{\mathscr{A}_1}]$, by definition we know that $(\omega(a), \omega(c))$ with a label $\omega(\circ_1)\omega(b)$ is an edge in $G^L[\widetilde{\mathscr{A}_2}]$. Whence,

$$\omega(a \circ_1 b) = \omega(a)\omega(\circ_1)\omega(b),$$

i.e., $\omega : \widetilde{\mathscr{A}_1} \to \widetilde{\mathscr{A}_2}$ is an isomorphism from $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$ to $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$. □

Generally, let $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ be two algebraic multi-systems associated with labeled graphs $G^L[\widetilde{\mathscr{A}_1}], G^L[\widetilde{\mathscr{A}_2}]$. A *homomorphism* $\iota : G^L[\widetilde{\mathscr{A}_1}] \to G^L[\widetilde{\mathscr{A}_2}]$ is a mapping $\iota : V(G^L[\widetilde{\mathscr{A}_1}]) \to V(G^L[\widetilde{\mathscr{A}_2}])$ and $\iota : \widetilde{\mathscr{O}}_1 \to \widetilde{\mathscr{O}}_2$ such that $\iota(a, c) = (\iota(a), \iota(c))$ with a label $\iota(\circ)\iota(b)$ for $\forall (a, c) \in E(G^L[\widetilde{\mathscr{A}_1}])$ with a label $\circ b$. We get a result on homomorphisms of labeled graphs following.

Theorem 7.4 *Let $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$, $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ be algebraic multi-systems, where $\widetilde{\mathscr{A}_k} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i^k$, $\widetilde{\mathscr{O}}_k = \bigcup\limits_{i=1}^{m} \mathcal{O}_i^k$ for $k = 1, 2$ and $\iota : (\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1) \to (\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$ a homomorphism. Then there is a homomorphism $\iota : G^L[\mathscr{A}_1] \to G^L[\mathscr{A}_2]$ from $G^L[\mathscr{A}_1]$ to $G^L[\mathscr{A}_2]$ induced by ι.*

Proof By definition, we know that $o : V(G^L[\mathscr{A}_1]) \to V(G^L[\mathscr{A}_2])$. Now if $(a, c) \in E(G^L[\mathscr{A}_1])$ with a label $\circ b$, then there must be $a \circ b = c$ in $(\widetilde{\mathscr{A}_1};\widetilde{\mathscr{O}}_1)$. Hence, $\iota(a)\iota(\circ)\iota(b) = \iota(c)$ in $(\widetilde{\mathscr{A}_2};\widetilde{\mathscr{O}}_2)$, where $\iota(\circ) \in \widetilde{\mathscr{O}}_2$ by definition. Whence, $(\iota(a), \iota(c)) \in E(G^L[\mathscr{A}_1])$ with a label $\iota(\circ)\iota(b)$ in $G^L[\mathscr{A}_2]$, i.e., ι is a homomorphism between $G^L[\mathscr{A}_1]$ and $G^L[\mathscr{A})_2]$. Therefore, ι induced a homomorphism from $G^L[\mathscr{A}_1]$ to $G^L[\mathscr{A}_2]$. □

Notice that an algebraic multi-system $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ is a combinatorial system $\mathscr{C}_\Gamma$ with an underlying graph Γ, called a Γ-*multi-system*, where

$$V(\Gamma) = \{\mathscr{H}_i | 1 \le i \le m\},$$

$$E(\Gamma) = \{(\mathscr{H}_i, \mathscr{H}_j) | \exists a \in \mathscr{H}_i, b \in \mathscr{H}_j \text{ with } (a, b) \in E(G^L[\widetilde{\mathscr{A}}]) \text{ for } 1 \le i, j \le m\}.$$

We obtain conditions for an algebraic multi-system with a graphical structure in the following.

Theorem 7.5 *Let* $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ *be an algebraic multi-system. Then it is*

(i) a circuit multi-system if and only if there is arrangement $\mathscr{H}_{l_i}, 1\leq i\leq m$ *for* $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_m$ *such that*

$$\mathscr{H}_{l_{i-1}}\bigcap\mathscr{H}_{l_i}\neq\emptyset,\ \mathscr{H}_{l_i}\bigcap\mathscr{H}_{l_{i+1}}\neq\emptyset$$

for any integer $i(mod\ m),\ 1\leq i\leq m$ *but*

$$\mathscr{H}_{l_i}\bigcap\mathscr{H}_{l_j}=\emptyset$$

for integers $j\neq i-1,i,i+1(mod\ m)$;

(ii) a star multi-system if and only if there is arrangement $\mathscr{H}_{l_i}, 1\leq i\leq m$ *for* $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_m$ *such that*

$$\mathscr{H}_{l_1}\bigcap\mathscr{H}_{l_i}\neq\emptyset\ \ but\ \ \mathscr{H}_{l_i}\bigcap\mathscr{H}_{l_j}=\emptyset$$

for integers $1<i,j\leq m, i\neq j$.

(iii) a tree multi-system if and only if any subset of $\widetilde{\mathscr{A}}$ *is not a circuit multi-system under operations in* $\widetilde{\mathscr{O}}$.

Proof By definition, these conditions really ensure a circuit, star, or a tree multi-system. Conversely, a circuit, star, or a tree multi-system constrains these conditions, respectively. □

Now if an associative system $(\mathscr{A};\circ)$ has a unit and inverse element $a_\circ^{-1}$ for any element $a\in\mathscr{A}$, i.e., a group, then for any elements $x,y\in\mathscr{A}$, there is an edge $(x,y)\in E(G^L[\mathscr{A}])$. In fact, by definition, there is an element $z\in\mathscr{A}$ such that $x_\circ^{-1}\circ y=z$. Whence, $x\circ z=y$. By definition, there is an edge (x,y) with a label $\circ z$ in $G^L[\mathscr{A}]$, and an edge (y,x) with label $z_\circ^{-1}$. Thereafter, the diagram of a group is a complete graph attached with a loop at each vertex, denoted by $K[\mathscr{A};\circ]$. As a by-product, the diagram $G^L[\widetilde{G}]$ of a m-group $\widetilde{G}$ is a union of m complete graphs with the same vertices, each attached with m loops.

Summarizing previous discussion, we can sketch the diagram of a multi-group as follows.

Theorem 7.6 *Let* $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ *be a multi-group with* $\widetilde{\mathscr{A}}=\bigcup\limits_{i=1}^{m}\mathscr{H}_i$, $\widetilde{\mathscr{O}}=\bigcup\limits_{i=1}^{m}\mathcal{O}_i$, $\mathcal{O}_i=\{\circ_{ij},1\leq j\leq n_i\}$ *and* $(\mathscr{H}_i;\circ_{ij})$ *a group for integers* $i,j,\ 1\leq i\leq m, 1\leq i\leq n_i$. *Then its diagram* $G^L[\mathscr{A}]$ *is*

$$G^L[\mathscr{A}]=\bigcup_{i=1}^{m}\bigcup_{j=1}^{n_i}K[\mathscr{H}_i;\circ_{ij}].$$

Corollary 7.1 *The diagram of a field* $(\mathscr{H};+,\circ)$ *is a union of two complete graphs attached with* 2 *loops at each vertex.*

Corollary 7.2 *Let* $(\widetilde{\mathscr{A}};\widetilde{\mathscr{O}})$ *be a multi-group. Then* $G^L[\mathscr{A}]$ *is hamiltonian if and only if* $\mathscr{C}_\Gamma$ *is hamiltonian.*

Proof Notice that $\mathscr{C}_\Gamma$ is an resultant graph in $G^L[\mathscr{A}]$ shrinking each $\bigcup\limits_{j=1}^{n_i} K[\mathscr{H}_i;\circ_{ij}]$ to a vertex $\mathscr{H}_i$ for $1 \leq i \leq m$ by definition. Whence, $\mathscr{C}_\Gamma$ is hamiltonian if $G^L[\mathscr{A}]$ is hamiltonian.

Conversely, if $\mathscr{C}_\Gamma$ is hamiltonian, we can easily find a hamiltonian circuit in $G^L[\mathscr{A}]$ by applying Theorem 7.6. □

§8. Remarks

8.1 These conceptions of multi-group, multi-ring, multi-field and multi-vector space are first presented in [11]-[14] introduced by Smarandache multi-spaces. In Sections $4-5$, we consider their general case, i.e., *multi-operation systems* and extend the homomorphism theorem to this multi-system. Section 6 is a generalization of works in [13] to multi-modules. There are many trends or topics in multi-systems should be researched, such as extending those of results in groups, rings or linear spaces to multi-systems.

8.2 The topic discussed in Section 7 can be seen as an application of combinatorial speculation([16]) to classical algebra. In fact, there are many research trends in *combinatorially algebraic systems*, in algebra or combinatorics. For example, *given an underlying combinatorial structure G, what can we say about its algebraic behavior*? Similarly, *what can we know on its graphical structure, such as in what condition it has a hamiltonian circuit, or a* 1*-factor*? *When it is regular*? $\cdots$, etc..

References

[1] G.Birkhoff and S.MacLane, *A Survey of Modern Algebra* (4th edition), Macmillan Publishing Co., Inc, 1977.

[2] G.Chartrand and L.Lesniak, *Graphs & Digraphs*, Wadsworth, Inc., California, 1986.

[3] J.E.Graver and M.E.Watkins, *Combinatorics with Emphasis on the Theory of Graphs*, Springer-Verlag, New York Inc,1977.

[4] L.Kuciuk and M.Antholy, An Introduction to Smarandache Geometries, *JP Journal of Geometry and Topology*, 5(1), 2005,77-81.

[5] J.C.Lu, *Fangfo Analyzing LAO ZHI - Explaining TAO TEH KING by TAI JI* (in Chinese), Tuan Jie Publisher, Beijing, 2004.

[6] J.C.Lu, *Originality's Come - Fangfo Explaining the Vajra Paramita Sutra* (in Chinese), Beijing Tuan Jie Publisher, 2004.

[7] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[8] L.F.Mao, On automorphism groups of maps, surfaces and Smarandache geometries, *Scientia Magna*, Vol.1(2005), No.2,55-73.

[9] L.F.Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[10] L.F.Mao, *Selected Papers on Mathematical Combinatorics*, World Academic Union, 2006.

[11] L.F.Mao, On algebraic multi-group spaces, *Scientia Magna*, Vol.2,No.1(2006), 64-70.

[12] L.F.Mao, On multi-metric spaces, *Scientia Magna*, Vol.2,No.1(2006), 87-94.

[13] L.F.Mao, On algebraic multi-vector spaces, *Scientia Magna*, Vol.2,No.2(2006), 1-6.

[14] L.F.Mao, On algebraic multi-ring spaces, *Scientia Magna*, Vol.2,No.2(2006), 48-54.

[15] L.F.Mao, Smarandache multi-spaces with related mathematical combinatorics, in Yi Yuan and Kang Xiaoyu ed: *Research on Smarandache Problems*, High American Press, 2006.

[16] L.F.Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[17] L.F.Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[18] L.Z.Nie and S.S.Ding, *Introduction to Algebra* (in Chinese), Higher Education Publishing Press, 1994.

[19] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[20] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[22] F.Smarandache, V.Christianto, Fu Yuhua, R.Khrapko and J.Hutchison, *Unfolding the Labyrinth: Open Problems in Physics, Mathematics, Astrophysics and Other Areas of Science*, Hexis, Phoenix, 2006.

[23] M.Tegmark, Parallel Universes, in *Science and Ultimate Reality: From Quantum to Cosmos*, ed. by J.D.Barrow, P.C.W.Davies and C.L.Harper, Cambridge University Press, 2003.

[24] M.Y.Xu, *Introduction to Group Theory* (in Chinese)(I)(II), Science Press, Beijing, 1999.

Math.Combin.Book.Ser. Vol.3 (2008), 28-33

A Double Cryptography Using the Smarandache Keedwell Cross Inverse Quasigroup

Tèmítọ́pẹ́ Gbọ́láhàn Jaíyéọlá

(Department of Mathematics of Obafemi Awolowo University, Ile Ife, Nigeria.)

E-mail: tjayeola@oauife.edu.ng

Abstract: The present study further strengthens the use of the Keedwell CIPQ against attack on a system by the use of the Smarandache Keedwell CIPQ for cryptography in a similar spirit in which the cross inverse property has been used by Keedwell. This is done as follows. By constructing two S-isotopic S-quasigroups(loops) U and V such that their Smarandache automorphism groups are not trivial, it is shown that U is a SCIPQ(SCIPL) if and only if V is a SCIPQ(SCIPL). Explanations and procedures are given on how these SCIPQs can be used to double encrypt information.

Key Words: Smarandache holomorph of loops, Smarandache cross inverse property quasigroups(CIPQs), Smarandache automorphism group, cryptography.

AMS(2000): 20NO5, 08A05.

§1. Introduction

1.1 Quasigroups and Loops

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ;

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. To every loop $(L, \cdot)$ with automorphism group $AUM(L, \cdot)$, there corresponds another loop. Let the set $H = (L, \cdot) \times AUM(L, \cdot)$. If we define '$\circ$' on H such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H$, then $H(L, \cdot) = (H, \circ)$ is a loop as shown in Bruck [6] and is called the Holomorph of $(L, \cdot)$.

A loop is a weak inverse property loop(WIPL) if and only if it obeys the identity

$$x(yx)^\rho = y^\rho \qquad \text{or} \qquad (xy)^\lambda x = y^\lambda.$$

[1]Received May 6, 2008. Accepted August 18, 2008.

A loop(quasigroup) is a cross inverse property loop(quasigroup)[CIPL(CIPQ)] if and only if it obeys the identity

$$xy \cdot x^\rho = y \qquad \text{or} \qquad x \cdot yx^\rho = y \qquad \text{or} \qquad x^\lambda \cdot (yx) = y \qquad \text{or} \qquad x^\lambda y \cdot x = y.$$

A loop(quasigroup) is an automorphic inverse property loop(quasigroup)[AIPL(AIPQ)] if and only if it obeys the identity

$$(xy)^\rho = x^\rho y^\rho \text{ or } (xy)^\lambda = x^\lambda y^\lambda.$$

The set $SYM(G,\cdot) = SYM(G)$ of all bijections in a groupoid $(G,\cdot)$ forms a group called the permutation(symmetric) group of the groupoid $(G,\cdot)$. Consider $(G,\cdot)$ and $(H,\circ)$ been two distinct groupoids(quasigroups, loops). Let A, B and C be three distinct non-equal bijective mappings, that maps G onto H. The triple $\alpha = (A,B,C)$ is called an isotopism of $(G,\cdot)$ onto $(H,\circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall\ x, y \in G.$$

If $(G,\cdot) = (H,\circ)$, then the triple $\alpha = (A,B,C)$ of bijections on $(G,\cdot)$ is called an autotopism of the groupoid(quasigroup, loop) $(G,\cdot)$. Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid(quasigroup, loop) $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

As observed by Osborn [17], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [1]-[4], Belousov and Tzurkan [5] and recent studies of Keedwell [12], Keedwell and Shcherbacov [13]-[15] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations(i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography.

Interestingly, Huthnance [7] showed that if $(L,\cdot)$ is a loop with holomorph $(H,\circ)$, $(L,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL. But the holomorphic structure of AIPL and a CIPL has just been revealed by Jaíyéọlá [11].

In the quest for the application of CIPQs with long inverse cycles to cryptography, Keedwell [12] constructed the following CIPQ which we shall specifically call Keedwell CIPQ.

Theorem 1.1 *Let $(G,\cdot)$ be an abelian group of order n such that $n+1$ is composite. Define a binary operation '$\circ$' on the elements of G by the relation $a \circ b = a^r b^s$, where $rs = n+1$. Then $(G,\circ)$ is a CIPQ and the right crossed inverse of the element a is a^u, where $u = (-r)^3$*

The author also gave examples and detailed explanation and procedures of the use of this CIPQ for cryptography. Cross inverse property quasigroups have been found appropriate for cryptography because of the fact that the left and right inverses x^λ and x^ρ of an element x do not coincide unlike in left and right inverse property loops, hence this gave rise to what is called *cycle of inverses* or *inverse cycles* or simply *cycles*, i.e finite sequence of elements $x_1, x_2, \cdots, x_n$ such that $x_k^\rho = x_{k+1} \mod n$. The number n is called the length of the cycle. The origin of the idea of cycles can be traced back to Artzy [1],[4] where he also found there existence in WIPLs apart form CIPLs. In his two papers, he proved some results on possibilities for the values of

n and for the number m of cycles of length n for WIPLs and especially CIPLs. We call these *Cycle Theorems* for now.

In application, it is assumed that the message to be transmitted can be represented as single element x of a quasigroup $(L, \cdot)$ and that this is enciphered by multiplying by another element y of L so that the encoded message is yx. At the receiving end, the message is deciphered by multiplying by the right inverse y^ρ of y. If a left(right) inverse quasigroup is used and the left(right) inverse of x is x^λ (x^ρ), then the left(right) inverse of x^λ (x^ρ) is necessarily x. But if a CIPQ is used, this is not necessary the situation. This fact makes an attack on the system more difficult in the case of CIPQs.

1.2 Smarandache Quasigroups and Loops

The study of Smarandache loops was initiated by W. B. Vasantha Kandasamy in 2002. In her book [19], she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In [9], the present author defined a Smarandache quasigroup(S-quasigroup) to be a quasigroup with at least a non-trivial associative subquasigroup called a Smarandache subsemigroup (S-subsemigroup). Examples of Smarandache quasigroups are given in Muktibodh [16]. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [20], on the study of Smarandache notions in algebraic structures, she introduced Smarandachely left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. In [8], the present author introduced Smarandachely inverse property loops(IPL) and weak inverse property loops(WIPL).

A quasigroup(loop) is called a *Smarandache certain quasigroup(loop)* if it has at least a non-trivial subquasigroup(subloop) with the certain property and the latter is referred to as the *Smarandache certain subquasigroup(subloop)*. For example, a loop is called a *Smarandache CIPL(SCIPL)* if it has at least a non-trivial subloop that is a CIPL and the latter is referred to as the *Smarandache CIP-subloop*. By an *initial S-quasigroup* L with an initial S-subquasigroup L', we mean L and L' are pure quasigroups, i.e. they do not obey a certain property(not of any variety).

If L is a S-groupoid with a S-subsemigroup H, then the set $SSYM(L, \cdot) = SSYM(L)$ of all bijections A in L such that $A : H \to H$ forms a group called the *Smarandache permutation(symmetric) group of the S-groupoid*. In fact, $SSYM(L) \leqslant SYM(L)$.

Definition 1.1 *Let $(L, \cdot)$ and $(G, \circ)$ be two distinct groupoids that are isotopic under a triple (U, V, W). Now, if $(L, \cdot)$ and $(G, \circ)$ are S-groupoids with S-subsemigroups L' and G' respectively such that $A : L' \to G'$, where $A \in \{U, V, W\}$, then the isotopism $(U, V, W) : (L, \cdot) \to (G, \circ)$ is called a Smarandache isotopism(S-isotopism).*

Thus, if $U = V = W$, then U is called a Smarandache isomorphism. Hence we write $(L, \cdot) \succsim (G, \circ)$.

But if $(L, \cdot) = (G, \circ)$, then the autotopism (U, V, W) is called a Smarandache autotopism (S-autotopism) and they form a group $SAUT(L, \cdot)$ which will be called the Smarandache autotopism group of $(L, \cdot)$. Observe that $SAUT(L, \cdot) \leqslant AUT(L, \cdot)$. Furthermore, if $U = V = W$, then U is called a Smarandache automorphism of $(L, \cdot)$. Such Smarandache permutations form a group

$SAUM(L,\cdot)$ *called the Smarandache automorphism group(SAG) of* $(L,\cdot)$.

Now, set $H_S = (L,\cdot) \times SAUM(L,\cdot)$. If we define '$\circ$' on H_S such that $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all $(\alpha, x), (\beta, y) \in H_S$, then $H_S(L,\cdot) = (H_S, \circ)$ is a S-quasigroup(S-loop) with S-subgroup $(H', \circ)$ where $H' = L' \times SAUM(L)$ and thus will be called the *Smarandache Holomorph(SH)* of $(L,\cdot)$.

The aim of the present study is to further strengthen the use of the Keedwell CIPQ against attack on a system by the use of the Smarandache Keedwell CIPQ for cryptography in a similar spirit in which the cross inverse property has been used by Keedwell. This is done as follows. By constructing two S-isotopic S-quasigroups(loops) U and V such that their Smarandache automorphism groups are not trivial, it is shown that U is a SCIPQ(SCIPL) if and only if V is a SCIPQ(SCIPL). Explanations and procedures are given on how these SCIPQs can be used to double encrypt information.

§2. Preliminary Results

Definition 2.1(Smarandachely Keedwell CIPQ) *Let Q be an initial S-quasigroup with an initial S-subquasigroup P. Q is called a Smarandachely Keedwell CIPQ(SKCIPQ) if P is isomorphic to the Keedwell CIPQ, say under a mapping ϕ.*

The following results that have recently been established are of paramount importance to prove the main result in this paper.

Theorem 2.1(Jaíyéọlá [10]) *Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups such that $SAUM(U)$ and $SAUM(V)$ are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $H_S(U) \succsim H_S(V)$ if and only if $x\delta \otimes y\gamma = (x\beta \oplus y)\delta\ \forall\ x, y \in L,\ \beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$.*

Theorem 2.2(Jaíyéọlá [11]) *The holomorph $H(L)$ of a quasigroup(loop) L is a Smarandache CIPQ(CIPL) if and only if $SAUM(L) = \{I\}$ and L is a Smarandache CIPQ(CIPL).*

§3. Main Result with Applications

3.1 Main result

Theorem 3.1 *Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups(S-loops) that are S-isotopic under the triple of the form $(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1})$ for all $\beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$ such that their Smarandache automorphism groups are non-trivial and are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, U is a SCIPQ(SCIPL) if and only if V is a SCIPQ(SCIPL).*

Proof Following Theorem 2.1, $H_S(U) \succsim H_S(V)$. Also, by Theorem 2.2, $H_S(U)(H_S(V))$ is a SCIPQ (SCIPL) if and only if $SAUM(U) = \{I\}(SAUM(V) = \{I\})$ and $U(V)$ is a

SCIPQ(SCIPL).

Now let U be an SCIPQ(SCIPL), then since $H_S(U)$ has a subquasigroup(subloop) that is isomorphic to a S-CIP-subquasigroup(subloop) of U and that subquasigroup (subloop) is isomorphic to a S-subquasigroup(subloop) of $H_S(V)$ which is isomorphic to a S-subquasigroup (subloop) of V, V is a SCIPQ(SCIPL). The proof for the converse is similar. $\square$

3.2 Application To Cryptography

Let the Smarandache Keedwell CIPQ be the SCIPQ U in Theorem 3.1. Definitely, its Smarandache automorphism group is non-trivial because as shown in Theorem 2.1 of Keedwell [12]. For any CIPQ, the mapping $J_\rho : x \to x^\rho$ is an automorphism. This mapping will be trivial only if the S-CIP-subquasigroup of U is unipotent. For instance, in Example 2.1 of Keedwell [12], the CIPQ $(G, \circ)$ obtained is unipotent because it was constructed using the cyclic group $C_5 =< c : c^5 = e >$ and defined as $a \circ b = a^3b^2$. But in Example 2.2, the CIPQ gotten is not unipotent as a result of using the cyclic group $C_{11} =< c : c^{11} = e >$. Thus, the choice of a Smarandache Keedwell CIPQ which suits our purpose in this work for a cyclic group of order n is one in which $rs = n + 1$ and $r + s \neq n$. Now that we have seen a sample for the choice of U, the initial S-quasigroup V can then be obtained as shown in Theorem 3.1. By Theorem 3.1, V is a SCIPQ.

Now, according to Theorem 2.1, by the choice of the mappings $\alpha, \beta \in SAUM(U)$ and $\psi \in SSYM(L)$ to get the mappings δ, γ, a SCIPQ V can be produced following Theorem 3.1. So, the secret keys for the systems are $\{\alpha, \beta, \psi, \phi\} \equiv \{\delta, \gamma, \phi\}$. Thus whenever a set of information or messages is to be transmitted, the sender will enciphere in the Smarandache Keedwell CIPQ by using specifically the S-CIP-subquasigroup in it(as described earlier on in the introduction) and then enciphere again with $\{\alpha, \beta, \psi, \phi\} \equiv \{\delta, \gamma, \phi\}$ to get a SCIPQ V which is the set of encoded messages. At the receiving end, the message V is deciphered by using an inverse isotopism(i.e inverse key of $\{\alpha, \beta, \psi\} \equiv \{\delta, \gamma\}$) to get U and then deciphere again(as described earlier on in the introduction) to get the messages. The secret key can be changed over time. The method described above is a double encryption and its a double protection. It protects each piece of information(element of the quasigroup) and protects the combined information(the quasigroup as a whole). Its like putting on a pair of socks and shoes or putting on under wears and clothes, the body gets better protection. An added advantage of the use of Smarandache Keedwell CIPQ over Keedwell CIPQ in double encryption is that the since the S-CIP-subquasigroups of the Smarandache Keedwell CIPQ in use could be more than one, then, the S-CIP-subquasigroups can be replaced overtime.

References

[1] R. Artzy, On loops with special property, *Proc. Amer. Math. Soc.* 6(1955), 448-453.

[2] R. Artzy, Crossed inverse and related loops, *Trans. Amer. Math. Soc.* 91, 3(1959), 480-492.

[3] R. Artzy, On Automorphic-Inverse Properties in Loops, *Proc. Amer. Math. Soc.* 10,4 (1959), 588-591.

[4] R. Artzy, Inverse-Cycles in Weak-Inverse Loops, *Proc. Amer. Math. Soc.* 68, 2(1978), 132-134.

[5] V. D. Belousov , Crossed inverse quasigroups(CI-quasigroups), *Izv. Vyss. Ucebn; Zaved. Matematika* 82(1969), 21-27.

[6] R. H. Bruck, Contributions to the theory of loops, *Trans. Amer. Math. Soc.* 55(1944), 245-354.

[7] E. D. Huthnance Jr., *A theory of generalised Moufang loops*, Ph.D. thesis, Georgia Institute of Technology, 1968.

[8] T. G. Jaíyéọlá, An holomorphic study of the Smarandache concept in loops, *Scientia Magna Journal*, 2, 1(2006), 1-8.

[9] T. G. Jaíyéọlá, Parastrophic invariance of Smarandache quasigroups, *Scientia Magna Journal*, 2, 3(2006), 48-53.

[10] T. G. Jaíyéọlá , A Pair Of Smarandachely Isotopic Quasigroups And Loops Of The Same Variety, *International J.Math. Combina.*, Vol.2,2008, 36-44.

[11] T. G. Jaíyéọlá, An Holomorphic Study Of Smarandache Automorphic and Cross Inverse Property Loops, Proceedings of the 4^{th} International Conference on Number Theory and Smarandache Problems, *Scientia Magna Journal*, Vol. 4, No. 1(2008), 102-108.

[12] A. D. Keedwell, Crossed-inverse quasigroups with long inverse cycles and applications to cryptography, *Australas. J. Combin.*, 20 (1999), 241-250.

[13] A. D. Keedwell and V. A. Shcherbacov, On m-inverse loops and quasigroups with a long inverse cycle, *Australas. J. Combin.*, 26(2002), 99-119.

[14] A. D. Keedwell and V. A. Shcherbacov , Construction and properties of (r, s, t)-inverse quasigroups I, *Discrete Math.*, 266(2003), 275-291.

[15] A. D. Keedwell and V. A. Shcherbacov, Construction and properties of (r, s, t)-inverse quasigroups II, *Discrete Math.*, 288 (2004), 61-71.

[16] A. S. Muktibodh, Smarandache Quasigroups, *Scientia Magna Journal*, 2, 1(2006), 13-19.

[17] J. M. Osborn, Loops with the weak inverse property, *Pac. J. Math.*, 10(1961), 295-304.

[18] Y. T. Oyebo and O. J. Adeniran, On the holomorph of central loops, Pre-print.

[19] W. B. Vasantha Kandasamy , *Smarandache Loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 2002, 128pp.

[20] W. B. Vasantha Kandasamy, Smarandache Loops, *Smarandache notions journal*, 13(2002), 252-258.

Math.Combin.Book.Ser. Vol.3 (2008), 34-39

On the Time-like Curves of Constant Breadth in Minkowski 3-Space

Süha Yılmaz and Melih Turgut

(Department of Mathematics of Buca Educational Faculty of Dokuz Eylül University, 35160 Buca-Izmir,Turkey.)

E-mail: suha.yilmaz@yahoo.com, melih.turgut@gmail.com

Abstract: A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache breadth curve*. In this paper, we study a special case of Smarandache breadth curves. Some characterizations of the time-like curves of constant breadth in Minkowski 3-Space are presented.

Key Words: Smarandache breadth curves, curves of constant breadth, Minkowski 3-Space, time-like curves.

AMS(2000): 51B20, 53C50.

§1. Introduction

Curves of constant breadth were introduced by L. Euler [3]. In [8], some geometric properties of plane curves of constant breadth are given. And, in another work [9], these properties are studied in the Euclidean 3-Space E^3. Moreover, M. Fujivara [5] had obtained a problem to determine whether there exist space curve of constant breadth or not, and he defined *breadth* for space curves and obtained these curves on a surface of constant breadth. In [1], this kind curves are studied in four dimensional Euclidean space E^4.

A regular curve with more than 2 breadths in Minkowski 3-space is called a *Smarandache breadth curve*. In this paper, we study a special case of Smarandache breadth curves. We investigate position vector of simple closed time-like curves and some characterizations in the case of constant breadth. Thus, we extended this classical topic to the space E_1^3, which is related with Smarandache geometries, see [4] for details. We used the method of [9].

§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented. A more complete elementary treatment can be found in the reference [2].

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

[1]Received July 1, 2008. Accepted August 25, 2008.

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be space-like if $\langle v, v \rangle > 0$ or $v = 0$, time-like if $\langle v, v \rangle < 0$ and null if $\langle v, v \rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors φ' are respectively space-like, time-like or null (light-like), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = \pm 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $\langle v, w \rangle = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E_1^3, the following Frenet formulae are given in [6]:

Let φ be a time-like curve, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\langle T, T \rangle = -1,\ \langle N, N \rangle = \langle B, B \rangle = 1,$$
$$\langle T, N \rangle = \langle T, B \rangle = \langle T, N \rangle = \langle N, B \rangle = 0$$

Let a and b be two time-like vectors in E_1^3. If a and b aren't in the same time cone then there is unique real number $\delta \geq 0$ called the hyperbolic angle between a and b, such that $g(a, b) = \|a\| \, \|b\| \cosh \delta$. Let $\vartheta = \vartheta(s)$ be a time-like curve in E_1^3. If tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called an inclined curve.

In [7], the author wrote a characterization for the inclined time-like curves with the following theorem.

Theorem 2.1 *Let $\varphi = \varphi(s)$ be an unit speed time-like curve in E_1^3. φ is an inclined curve if and only if*

$$\frac{\kappa}{\tau} = \text{constant}. \tag{2}$$

§3. The Time-like Curves of Constant Breadth in $\mathbf{E}_1^3$

Definition 3.1 *A regular curve with more than 2 breadths in Minkowski 3-space is called a Smarandache breadth curve.*

Let $\varphi = \varphi(s)$ be a Smarandache breadth curve. Moreover, let us suppose $\varphi = \varphi(s)$ simple closed time-like curve in the space E_1^3. These curves will be denoted by (C). The normal plane

at every point P on the curve meets the curve at a single point Q other than P. We call the point Q the opposite point of P. We consider a curve in the class Γ as in [?] having parallel tangents T and T^* in opposite directions at the opposite points φ and φ^* of the curve. A simple closed curve having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^*(s) = \varphi(s) + m_1 T + m_2 N + m_3 B, \tag{3}$$

where $m_i(s)$, $1 \leq i \leq 3$ are arbitrary functions and φ and φ^* are opposite points. Differentiating both sides of (3) and considering Frenet equations, we have

$$\left\{ \begin{array}{c} \dfrac{d\varphi^*}{ds} = T^* \dfrac{ds^*}{ds} = (\dfrac{dm_1}{ds} + m_2\kappa + 1)T + \\ (\dfrac{dm_2}{ds} + m_1\kappa - m_3\tau)N + (\dfrac{dm_3}{ds} + m_2\tau)B \end{array} \right\}. \tag{4}$$

Since $T^* = -T$. Rewriting (4), we have respectively,

$$\left\{ \begin{array}{c} \dfrac{dm_1}{ds} = -m_2\kappa - 1 - \dfrac{ds^*}{ds} \\ \dfrac{dm_2}{ds} = -m_1\kappa + m_3\tau \\ \dfrac{dm_3}{ds} = -m_2\tau \end{array} \right\}. \tag{5}$$

If we call ϕ as the angle between the tangent of the curve (C) at point $\varphi(s)$ with a given fixed direction and consider $\dfrac{d\phi}{ds} = \kappa$, we have (5) as follow:

$$\left\{ \begin{array}{c} \dfrac{dm_1}{d\phi} = -m_2 - f(\phi) \\ \dfrac{dm_2}{d\phi} = -m_1 + m_3\rho\tau \\ \dfrac{dm_3}{d\phi} = -m_2\rho\tau \end{array} \right\}, \tag{6}$$

where $f(\phi) = \rho + \rho^*$, $\rho = \dfrac{1}{\kappa}$ and $\rho^* = \dfrac{1}{\kappa^*}$ denote the radius of curvatures at φ and φ^*, respectively. And using system (6), we have the following differential equation with respect to m_1 as

$$\frac{\kappa}{\tau}\left[\frac{d^3 m_1}{d\phi^3} + \frac{d^2 f}{d\phi^2}\right] + \frac{d}{d\phi}(\frac{\kappa}{\tau})\left[\frac{d^2 m_1}{d\phi^2} - m_1 + \frac{df}{d\phi}\right] + (\frac{\tau^2 - \kappa^2}{\tau\kappa})\frac{dm_1}{d\phi} + \frac{\tau}{\kappa} f = 0. \tag{7}$$

Equation (7) is a characterization for φ^*. If the distance between opposite points of (C) and (C^*) is constant, then, we can write that

$$\|\varphi^* - \varphi\| = -m_1^2 + m_2^2 + m_3^2 = l^2 = constant. \tag{8}$$

Hence, we write

$$-m_1\frac{dm_1}{d\phi} + m_2\frac{dm_2}{d\phi} + m_3\frac{dm_3}{d\phi} = 0. \tag{9}$$

Considering system (6), we obtain

$$m_1(\frac{dm_1}{d\phi} + m_2) = 0. \tag{10}$$

We write $m_1 = 0$ or $\dfrac{dm_1}{d\phi} = -m_2$. Thus, we shall study in the following subcases.

Case 1. $\dfrac{dm_1}{d\phi} = -m_2$. Then $f(\phi) = 0$. In this case, (C^*) is translated by the constant vector

$$u = m_1 T + m_2 N + m_3 B \tag{11}$$

of (C). Now, let us to investigate solution of the equation (7), in some special cases.

Case 1.1 Suppose that φ is an inclined curve. If we rewrite (7), we have the following differential equation:

$$\frac{d^3 m_1}{d\phi^3} + (\frac{\tau^2}{\kappa^2} - 1)\frac{dm_1}{d\phi} = 0. \tag{12}$$

General solution of (12) depends on character of $\dfrac{\tau}{\kappa}$. Due to this, we distinguish following subcases.

Case 1.1.1 $\tau > \kappa$. Then the solution above differential equation is:

$$m_1 = C_1 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi + C_2 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi. \tag{13}$$

And therefore, we have m_2 and m_3, respectively,

$$m_2 = \sqrt{\frac{\tau^2}{\kappa^2} - 1}\left\{C_1 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi - C_2 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi\right\}, \tag{14}$$

$$m_3 = \frac{\tau}{\kappa}\left[C_1 \cos\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi + C_2 \sin\sqrt{\frac{\tau^2}{\kappa^2} - 1}\phi\right]. \tag{15}$$

where C_1 and C_2 are real numbers.

Case 1.1.2 $\tau < \kappa$. Then the solution has the form

$$m_1 = A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi.} \tag{16}$$

Hence, we have m_2 and m_3 as follows:

$$m_2 = \sqrt{1 - \frac{\tau^2}{\kappa^2}}\left\{-A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi}\right\}, \tag{17}$$

$$m_3 = \frac{\tau}{\kappa}\left[A_1 e^{\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi} + A_2 e^{-\sqrt{1-\frac{\tau^2}{\kappa^2}}\phi}\right]. \tag{18}$$

where A_1 and A_2 are real numbers.

Corollary 3.1 *Position vector of φ^* can be formed by the equations* (13), (14) *and* (15) *or* (16), (17) *and* (18) *according to ratio of* $\dfrac{\tau}{\kappa}$.

Case 1.2 Let us suppose $m_1 = c_1 = constant \neq 0$. Thus $m_2 = 0$. From $(6)_3$ we easily have $m_3 = c_3 = constant$. And using $(6)_2$ we get

$$\frac{\kappa}{\tau} = \frac{c_3}{c_1} = constant. \tag{19}$$

Equation (19) shows that φ is an inclined curve. Therefore, **Case 1.2** is a characterization for the inclined time-like curves of constant breadth in E_1^3. Then the position vector of φ^* can be written as follow:

$$\varphi^* = \varphi + c_1 T + c_3 B. \tag{20}$$

And curvature of φ^* is obtained as

$$\kappa^* = \kappa. \tag{21}$$

Case 2 $m_1 = 0$. Then $m_2 = -f(\phi)$. And, here, let us suppose that φ is an inclined curve. Thus, the equation (7) has the form

$$\frac{d^2 f}{d\phi^2} + \frac{\tau^2}{\kappa^2} f = 0. \tag{22}$$

The solution of (22) is

$$f(\phi) = L_1 \cos \frac{\tau}{\kappa}\phi + L_2 \sin \frac{\tau}{\kappa}\phi. \tag{23}$$

where L_1 and L_2 are real numbers. Using equation (23), we have m_2 and m_3

$$m_2 = -L_1 \cos \frac{\tau}{\kappa}\phi - L_2 \sin \frac{\tau}{\kappa}\phi = -\rho - \rho^*, \tag{24}$$

$$m_3 = L_1 \sin \frac{\tau}{\kappa}\phi - L_2 \sin \frac{\tau}{\kappa}\phi. \tag{25}$$

And therefore, we write the position vector and the curvature of φ^*

$$\varphi^* = \varphi + (-\rho - \rho^*)N + (L_1 \sin \frac{\tau}{\kappa}\phi - L_2 \sin \frac{\tau}{\kappa}\phi)B, \tag{26}$$

$$\kappa^* = \frac{1}{L_1 \cos \frac{\tau}{\kappa}\phi + L_2 \sin \frac{\tau}{\kappa}\phi - \frac{1}{\kappa}}. \tag{27}$$

And the distance between the opposite points of (C) and (C^*) is

$$\|\varphi^* - \varphi\| = L_1^2 + L_2^2 = constant. \tag{28}$$

References

[1] A. Mağden and Ö. Köse, On The Curves of Constant Breadth, *Tr. J. of Mathematics*, **1** (1997), 277-284.

[2] B. O'Neill, *Semi-Riemannian Geometry,*, Academic Press, New York,1983.

[3] L. Euler, De Curvis Trangularibis, *Acta Acad. Petropol* (1780), 3-30.

[4] L. F. Mao, Pseudo-manifold geometries with applications, *International J.Math. Comb.*, Vol.1(2007), No.1, 45-58.

[5] M. Fujivara, *On Space Curves of Constant Breadth*, Tohoku Math. J. **5** (1914), 179-184.

[6] M. Petrovic-Torgasev and E. Sucurovic, Some characterizations of the spacelike, the timelike and the null curves on the pseudohyperbolic space H_0^2 in E_1^3, *Kragujevac J. Math.* **22** (2000), 71-82.

[7] N. Ekmekci, *The Inclined Curves on Lorentzian Manifolds*, Dissertation, Ankara University (1991)

[8] Ö. Köse, Some Properties of Ovals and Curves of Constant Width in a Plane, *Doga Mat.*,(8) **2** (1984), 119-126.

[9] Ö. Köse, On Space Curves of Constant Breadth, *Doga Math.* (10) **1** (1986), 11-14.

Math.Combin.Book.Ser. Vol.3 (2008), 40-50

On the Basis Number of the Strong Product of Theta Graphs with Cycles

[1]M.M.M. Jaradat, [2]M.F. Janem and [2]A.J. Alawneh

(1.Department of Mathematics and physics of Qata University, Doha-Qatar.)

(2.Department of Mathematics and Statistics of Jordan University of Science and Technology, Irbid-Jordan.)

E-mail: mmjst4@qu.edu.qa, janemajdah@yahoo.com, ameen@just.edu.jo

Abstract: A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph G is called a d-fold if each edge of G occurs in at most d of the cycles in the basis $\mathcal{B}$. A basis $\mathcal{B}$ for the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$. The basis number, $b(G)$, of a graph G is defined to be the least integer d such that there is a d-fold basis of the cycle space of G. MacLane [20] made a connection between the the number of occurrence of edges of a graph in its cycle bases and the planarity of a graph, which is related with parallel bundles on planar map geometries, a kind of Smarandache geometries. In fact, he proved that a graph G is planar if and only if $b(G) \leq 2$. Jaradat [10] gave an upper bound of the basis number of the strong product of a graph with a bipartite graph in terms of the factors. In this work, we show that the basis number of the strong product of a theta graph with a cycle is either 3 or 4. Our result, improves Jaradat's upper bound in the case of specializing the factors by a theta graph and a cycle.

Key Words: Cycle space, cycle basis, Smarandache basis, basis number, strong product.

AMS(2000): 05C38, 05C75.

§1. Introduction

In graph theory, there are many numbers that give rise to a better understanding and interpretation of the geometric properties of a given graph such as the crossing number, the thickness, the genus, the basis number, etc.. The basis number of a graph is of a particular importance because MacLane, in [20], made a connection between the number of occurrences of edges of a graph in its cycle bases and the planarity of a graph; in fact, he proved that a graph is planar if and only if its basis number is at most 2. For the completeness, it should be mentioned that a basis $\mathcal{B}$ of the cycle space $\mathcal{C}(G)$ of a graph G is Smarandachely if each edge of G occurs in at least 2 of the cycles in $\mathcal{B}$

Product of graphs occur naturally in discrete mathematics as tools in combinatorial constructions. They give rise to important classes of graphs and deep structure problems. There are many graph products in the literature, such as, Cartesian product, strong product, lexicographic product, semi-strong product and semi-composition product. The extensive literature

[1]Received July 18, 2008. Accepted August 30, 2008.

on products that has evolved over the years presents a wealth of profound and beautiful results. This led Imrich and Klavzar to write a whole book on graph products [7].

The main purpose of this paper is to investigate the basis number of the strong product of a theta graph with a cycle. Our result improves the upper bounds that expected from applying Jaradat's theorems.

§2. Definitions and preliminaries

Unless otherwise specified, the graphs considered in this paper are finite, undirected, simple and connected. For a given graph G, we denote the vertex set of G by $V(G)$ and the edge set by $E(G)$.

For a given graph G, the set $\mathcal{E}$ of all subsets of $E(G)$ forms an $|E(G)|$-dimensional vector space over Z_2 with vector addition $X \oplus Y = (X\backslash Y) \cup (Y \backslash X)$ and scalar multiplication $1 \cdot X = X$ and $0 \cdot X = \emptyset$ for all $X, Y \in \mathcal{E}$. The cycle space, $\mathcal{C}(G)$, of a graph G is the vector subspace of $(\mathcal{E}, \oplus, \cdot)$ spanned by the cycles of G. Note that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that for a connected graph G the dimension of the cycle space is the *cyclomatic number* or the *first Betti number*

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r \tag{1}$$

where r is the number of components in G.

The first important use of the basis number dates back to MacLane [20] when he made the connection between the basis number of a graph and the planarity. There after, in 1981, E. Schmeichel [21] formalized the definition of the basis number of a graph as follows: A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of G. A cycle basis $\mathcal{B}$ of G is called a d-fold if each edge of G occurs in at most d of the cycles in $\mathcal{B}$. *The basis number,* $b(G)$, of G is the least non-negative integer d such that $\mathcal{C}(G)$ has a *d-fold* basis.

Latter on, Schmeichel [21] investigate the basis number of the known classes of graphs such as the complete graphs K_n and the complete bipartite graphs $K_{n,m}$. In fact, he proved that $b(K_n) = 3$, for $n \geq 5$ and $b(K_{n,m}) = 4$ for all $n, m \geq 5$ except a few numbers of graphs. Also, he proved that for any positive integer r, there exists a graph G with $b(G) \geq r$. After that, he joined Banks to prove that the basis number of n-cube is 4 for all $n \geq 7$ (see [6])

Since 1992, many researchers were attracted to study the basis number of graph products. The Cartesian product, $\Box$, was studied by Ali and Marougi [3] when they gave the following result:

Theorem 2.1 (Ali and Marougi) *If G and H are two connected disjoint graphs, then* $\mathrm{b}(G \Box H) \leq \max\{\mathrm{b}(G) + \triangle(T_H), \ \mathrm{b}(H) + \triangle(T_G)\}$ *where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of H and G.*

Also, Alsardary and Wojciechowski [4] proved that for every $d \geq 1$ and $n \geq 2$, $b(K_n^d) \leq 9$ where K_n^d is a d times Cartesian product of the complete graph K_n.

Upper bounds of the strong product, $\boxtimes$, were obtained by Jaradat [11], [14] and [15] when he gave the following results:

Theorem 2.2(Jaradat) *Let G be a bipartite graph and H be a graph. Then* $b(H \boxtimes G) \leq \max\left\{b(G)+1, 2\Delta(G)+b(H)-1, \left\lfloor \frac{3\Delta(T_H)+1}{2} \right\rfloor, b(H)+2\right\}$.

Theorem 2.3(Jaradat) *Let G be a bipartite graph and C be a cycle. Then* $b(G \boxtimes C) \leq 4 + b(G)$.

The lexicographic product of two graphs G and H, $G[H]$, was studied by Jaradat and Al-zoubi [17] and Jaradat [13]. They obtained the following results:

Theorem 2.4 (Jaradat and Al-Zoubi) *For each two connected graphs G and H,* $b(G[H]) \leq \text{Max}\{4, 2\Delta(G) \; + b(H), 2 + b(G)\}$.

Theorem 2.5(Jaradat *Let G, T_1* and T_2 *be a graph, a spanning tree of G and a tree, respectively. Then,* $b(G[T_2]) \leq b(G[H]) \leq \max\left\{5, 4+ \; 2\Delta(T^G_{\min}) + b(H), 2 + b(G)\right\}$ *where T^G stands for the complement graph of a spanning tree T in G and $T_{\min}$ stands for a spanning tree for G such that* $\Delta(T^G_{\min}) = \min\{\; \Delta(T^G) | T$ *is a spanning tree of* $G\}$.

Ali [1], [2] gave an upper bound for the basis number of the semi-strong product, $\bullet$, and the direct product, $\times$, of some special graphs when he proved that $b(K_m \bullet K_n) \leq 9$ for any integers m, n and $b(C_n \times C_m) = 3$ for any two cycles C_n and C_m with $n, m \geq 3$. Also the following upper bound (among other results) were obtained by Jaradat [8], [9], [14] and [18]:

Theorem 2.6(Jaradat) *For each bipartite graphs G and H,* $b(G \times H) \leq 5 + b(G) + b(H)$.

Theorem 2.7 (Jaradat) *For each bipartite graphs G and H,*

$$b(G \bullet H) \leq \max\{b(G) + b(H) + \left\{\begin{array}{ll} 3, & \text{if both of } T_G \text{ and } T_H \text{ are paths,} \\ 4, & \text{if } T_H \text{ is a path,} \\ 5, & \text{if } T_G \text{ is a path,} \\ 6, & \text{if both of } T_G \text{ and } T_H \text{ are not paths.} \end{array}\right\}, \Delta(T_G) + b(H)\}$$

The wreath product, was studied by Jaradat and Al-Qeyyam (See [5], [12] and [16]).

For completeness, we recall that for any two graphs G and H, the strong product $G \boxtimes H$ is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$. The Cartesian product $G \square H$ is the graph with the vertex set $V(G \square H) = V(G) \times V(H)$ and the edge set $E(G \square H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2 = v_2$ or $u_1 = v_1$ and $u_2v_2 \in E(H)\}$. Also, the direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2) | u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$.

In the rest of this paper, $f_B(e)$ stand for the number of elements of B containing the edge e where $B \subseteq \mathcal{C}(G)$.

§3. The basis number of $\theta_n \boxtimes C_m$

In this section we investigate the basis number of the strong product of theta graphs and cycles. In fact we show that $3 \leq b(\theta_n \boxtimes C_m) \leq 4$. Throughout this section we assume that $1, 2, \ldots, n$ and $1, 2 \ldots, m$ to be the vertices of θ_n and C_m, respectively.

Definition 3.1 *A theta graph θ_n is defined to be a cycle to which we add a new edge that joins two non-adjacent vertices. We may assume 1 and δ are the two vertices of θ_n of degree 3.*

Applying Theorem 2.2 for the case $H = \theta_n$ and G to be a cycle of even length C_m, we get $b(\theta_n \boxtimes C_m) \leq \max\{2, 5, 3, 4\} = 5$. Also, applying the same theorem by considering $H = C_m$ and G to be a theta graph that contains no odd cycles θ_n, we get $b(C_m \boxtimes \theta_n) \leq \max\{3, 6, 3, 4\} = 6$. Moreover, by specializing G in Theorem 2.3 to θ_n that contains no odd cycle, then $b(\theta_n \boxtimes C_m) \leq 6$. However, these upper bounds will be reduced to 4 as we will see in Theorem 3.6. Now, for this purpose, we consider the following cycles: For each $j = 1, 2, \ldots, m-2$, set

$$\begin{aligned} \mathcal{A}_1^{(j)} &= (1,j)(2,j+1)(1,j+2)(\delta,j+1)(1,j), \\ \mathcal{A}_2^{(j)} &= (\delta,j)(\delta-1,j+1)(\delta,j+2)(1,j+1)\,(\delta,j)\,, \end{aligned}$$

and let

$$\mathcal{A}_1 = \bigcup_{j=1}^{m-2} \mathcal{A}_1^{(j)} \quad \text{and} \quad \mathcal{A}_2^{(i)} = \bigcup_{j=1}^{m-2} \mathcal{A}_2^{(j)}.$$

The following result will be useful in our main result.

Lemma 3.2 *Every linear combination of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2$ contains at least one edge of* $\{(1,j)(\delta,j+1), (1,j+1)(\delta,j) | 1 \leq j \leq m-2\}$.

Proof Consider $\mathcal{O}$ to be a linear combinations of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2^{(i)}$. Then

$$\mathcal{O} = \bigoplus_{j=1}^{s_1} \mathcal{A}_1^{(1_j)} \oplus \bigoplus_{j=1}^{s_2} \mathcal{A}_2^{(2_j)}$$

where $\mathcal{A}_1^{(1_j)} \in \mathcal{A}_1$, $\mathcal{A}_2^{(2_j)} \in \mathcal{A}_2$, $1_1 < 1_2 < \cdots < 1_{s_1}$ and $2_1 < 2_2 < \cdots < 2_{s_2}$. Now, let $t_1 = \min\{1_1, 2_1\}$. We now consider the following two cases.

Case 1. $t_1 = 1_1$. Then by the definition of $\mathcal{A}_1$, $\mathcal{A}_1^{(1_1)}$ contains the edge $(1, 1_1)(\delta, , 1_1 + 1)$ where $1_1 \leq m-2$. Since $E(\mathcal{A}_1^{(j)}) \cap E(\mathcal{A}_1^{(i)}) = \varnothing$, $(1, 1_1)(\delta, , 1_1 + 1) \notin \mathcal{A}_1^{(1_j)}$ for each $1 \leq j \leq s_1$. Also, since $1_1 \leq 2_1, (1, 1_1)(\delta, , 1_1 + 1) \notin \mathcal{A}_2^{(2_j)}$ for each $1 \leq j \leq s_2$. Therefore, $(1, 1_1)(\delta, , 1_1 + 1) \in \mathcal{O}$.

Case 2. $t_1 = 2_1$. Then we argue more or less as in Case 1, to have that $(1, 2_1 + 1)(\delta, , 2_1) \in \mathcal{O}$ where $2_1 \leq m-2$. □

Now, for $j = 1, 2, \ldots, m-1$, consider the following set of cycles:

$$\mathcal{K}_j = (1,j)(\delta,j)\,(1,j+1)\,(\delta,j+1)(1,j),$$

and let

$$\mathcal{K} = \bigcup_{j=1}^{m-1} \mathcal{K}_j.$$

Lemma 3.3 *Every linear combination of cycles of $\mathcal{K}$ contains at least one edge of* $\{(1,j)(\delta,j)|1 \leq j \leq m-1\}$.

Proof Let

$$\mathcal{O} = \sum_{i=1}^{s} \mathcal{K}_{j_i} (\mathrm{mod} 2)$$

where $\mathcal{K}_{j_i} \in \mathcal{K}$ and $j_1 < j_2 < \cdots < j_s \leq m-1$. Then by the definition of $\mathcal{K}$,

$$E(\mathcal{K}_{j_1}) \cap E(\cup_{i=2}^{s} \mathcal{K}_{j_i}) \subseteq \{(1, j_1+1)(\delta, j_1+1)\}.$$

But, $(1,j_1)(\delta,j_1) \in E(\mathcal{K}_{j_1})$. Hence, $(1,j_1)(\delta,j_1) \in \mathcal{O}$. $\square$

Lemma 3.4 Let θ_n be a graph of order $n \geq 4$ and C_m be a cycle of order $m \geq 3$. Then $b\,(\theta_n \boxtimes C_m) \geq 3$.

Proof Assume that $\theta_n \boxtimes C_m$ has a 2-fold basis $\mathcal{B}$. Since the girth of $\theta_n \boxtimes C_m$ is 3, we have that

$$\begin{aligned}
3\,|\mathcal{B}| &\leq 2\,|E(\theta_n \boxtimes C_m)| \\
3(3m(n+1)+1) &\leq 2(3m(n+1)+nm) \\
9mn+9m+3 &\leq 6mn+6m+2nm \\
mn+3m+3 &\leq 0 \\
m(n+3)+3 &\leq 0
\end{aligned}$$

which is a contradiction. Hence $\mathcal{B}(\theta_n \boxtimes C_m)$ is a 3-fold basis. $\square$

The following result of Jaradat and et al. will be needed in our coming result:

Proposition 3.5 (Jaradat and et al)*Let A and B be two linearly independent sets of cycles such that $E(A) \cap E(B)$ subset of an edge set of a forest or an empty set. Then $A \cup B$ is linearly independent.*

The following cycles which were introduced in [11] will be used frequently in the coming results.

$$
\begin{aligned}
\mathcal{L}_{ab} &= \Big\{\mathcal{L}^{(j)} = (a,v_j)(b,v_{j+1})(a,v_{j+1})(a,v_j) | j = 1,2,3,\cdots,m-1\Big\} \\
&\quad \cup \Big\{\mathcal{L}^{(n)} = (a,v_n)(b,v_1)(a,v_1)(a,v_n)\Big\} \\
\mathcal{T}_{ab} &= \Big\{\mathcal{T}^{(j)} = (a,v_j)(a,v_{j+1})(b,v_j)(a,v_j) | j = 1,2,3,\cdots,m-1\Big\} \\
&\quad \cup \Big\{\mathcal{T}^{(n)} = (a,v_n)(a,v_1)(b,v_n)(a,v_n)\Big\} \\
\mathcal{S}_{ab} &= \Big\{\mathcal{S}^{(j)} = (a,v_{j+1})(b,v_j)(b,v_{j+1})(a,v_{j+1}) | j = 1,2,3,\cdots,m-1\Big\} \\
&\quad \cup \Big\{\mathcal{S}^{(n)} = (a,v_1)(b,v_n)(b,v_1)(a,v_1)\Big\}.
\end{aligned}
$$

Also

$$
\mathcal{F}_n = \begin{cases} (a,v_1)(b,v_2)(a,v_3)(b,v_4)\ldots(a,v_{n-1})(b,v_n)(a,v_1) \text{ if } m \text{ is even,} \\ (a,v_1)(b,v_1)(a,v_2)(b,v_3)\ldots(a,v_{n-1})(b,v_n)(a,v_1) \text{ if } m \text{ is odd.} \end{cases}
$$

Let

$$
\mathcal{B}_{ab} = \mathcal{L}_{ab} \cup \mathcal{T}_{ab} \cup \mathcal{S}_{ab} \text{ and } \mathcal{B}^*_{ab} = \mathcal{B}_{ab} - \{\mathcal{S}^{(m)}\} \cup \{\mathcal{F}_m\}
$$

Moreover, by Theorem 2.6 of [11], we have that

$$
\dim \mathcal{C}(C_n \boxtimes C_m) = 3mn + 1. \tag{2}
$$

Note that $\theta_n \boxtimes C_m$ is decomposable into $(C_n \boxtimes C_m) \cup (1\alpha \Box N_m) \cup (1\alpha \times C_m)$ where N_m is the null graph with vertex set $V(C_m)$. Thus,

$$
\begin{aligned}
\dim \mathcal{C}(\theta_n \boxtimes C_m) &= \dim \mathcal{C}(C_n \boxtimes C_m) + m + 2m, \qquad (3) \\
&= 3mn + 3m + 1. \qquad (4)
\end{aligned}
$$

Now, we state and prove our main result.

Theorem 3.6 *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have* $3 \leq b(\theta_n \boxtimes C_m) \leq 4$.

Proof By Lemma 3.4, it is sufficient to exhibit a 4-fold basis, $\mathcal{B}$, for $\mathcal{C}(\theta_n \boxtimes C_m)$. According to the parity of m, n and δ (odd or even), we consider the following cases.

Case 1. m and n are even and δ is odd. Then define

$$
\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}}\right) \cup \mathcal{B}^*_{a_n a_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C\} \cup \{C_1, C_2, C_3, C_4, C_5\}
$$

where $B_{a_i a_{i+1}}$ and $B^*_{a_n a_1}$ are as in above and

$$
C = (1,1)(2,2)(3,1)(4,2)...(n-1,1)(n,2)(1,1).
$$

Also,

$$\begin{aligned}
C_1 &= (1,m-1)(2,m)(3,m)(4,m)...(\delta,m)\,(1,m-1)\,. \\
C_2 &= (1,1)(1,m)(\delta,m)(1,1). \\
C_3 &= (\delta,1)(\delta+1,2)(\delta,3)...(\delta,m-1)(1,m)\,(\delta,1)\,. \\
C_4 &= (1,1)\,(2,m)\,(3,1)\,(4,m)\,...\,(\delta,1)\,(1,m)\,(2,1)\,(3,m)\,...\,(\delta,m)\,(1,1)\,. \\
C_5 &= (1,m)(2,1)(3,m)(4,1)...(\delta,m)(1,m).
\end{aligned}$$

Let $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{a_i a_{i+1}} \cup \mathcal{B}^*_{a_n a_1} \cup \{C\}$. Note that $\mathcal{B}_1 =$B$(C_n \boxtimes C_m)$ is a basis for $\mathcal{C}(C_n \boxtimes C_m)$ (see Theorem 2.6, Case 1 of [11]). Thus, $\mathcal{B}_1$ is linearly independent. Note that C_5 contains the edge $(\delta,m)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1$. Hence, $\mathcal{B}_1 \cup \{C_5\}$ is linearly independent. Now, C_2 contains the edge $(\delta,m)(1,1)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_5\}$. So, $\mathcal{B}_1 \cup \{C_5, C_2\}$ is linearly independent. Similarly, the cycle C_4 contains the edge $(\delta,1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \cup \{C_5, C_2\}$. Thus, $\mathcal{B}_1 \cup \{C_2, C_4, C_5\}$ is linearly independent. Also, C_3 contains the edge $(\delta, m-1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_2, C_4, C_5\}$. Therefore, $\mathcal{B}_1 \cup \{C_2, C_3, C_4, C_5\}$ is linearly independent. Finally, C_1 contains the edge $(1,m-1)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_2, C_3, C_4, C_5\}$. Thus, $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. By Lemma 3.2, any linear combination of cycles of $\mathcal{A}_1 \cup \mathcal{A}_2$ contains at least one edge of $\{(1,j)(\delta,j+1),(1,j+1)(\delta,j) | 1 \le j \le m-2\}$ which does not occur in any cycle of $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$. Thus, $\mathcal{B}_1 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. Similarly, by Lemma 3.2, any linear combination of cycles of $\mathcal{K}$ contains at least one edge of $\{(1,j)(\delta,j) | 1 \le j \le m-1\}$, which does not occur in any cycle of $\mathcal{B}_1 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \{C_1, C_2, C_3, C_4, C_5\}$. Therefore, $\mathcal{B}(\theta_n \boxtimes C_m)$ is linearly independent. Note that

$$\begin{aligned}
|\mathcal{B}(\theta_n \boxtimes C_m)| &= |\mathcal{B}_1| + |\mathcal{K}| + |\mathcal{A}_1| + |\mathcal{A}_2| + \sum_{i=1}^{5} |C_i| \\
&= 3mn + 1 + |\mathcal{K}| + |\mathcal{A}_1| + |\mathcal{A}_2| + \sum_{i=1}^{5} |C_i| \\
&= 3mn + 1 + (m-1) + (m-2) + (m-2) + 5 \\
&= 3mn + 3m + 1 \\
&= 3m(n+1) + 1 \\
&= \dim \mathcal{C}(\theta_n \boxtimes C_m),
\end{aligned}$$

where the last equality follows from equation (4). Therefore, B$(\theta_n \boxtimes C_m)$ is a basis for $\mathcal{C}(\theta_n \boxtimes C_m)$. To complete the proof of this case, we show that B$(\theta_n \boxtimes C_m)$ is a 3-fold basis. Let $e \in E(\theta_n \boxtimes C_m)$. Then 1) if $e = (1,m-1)(2,m)$, then $f_{\mathcal{B}_1}(e) = 1$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 2) If $e \in \{(i,m)(i+1,m) | i = 2,3,\dots,m-1\}$, then $f_{\mathcal{B}_1}(e) = 2$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 3) If $e = (1,m)(\delta,m)$ or $(1,1)(\delta,1)$, then $f_{\mathcal{B}_1}(e) = 0$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 1$ and $f_{\{C_i\}}(e) \le 2$. 4) If $e = (1,1)(1,m)$, then $f_{\mathcal{B}_1}(e) = 2$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) = 1$. 5) If $e \in \{(i,1)(i+1,m),(i+1,1)(i,m) | i = 1,2,\dots,n-1\} \cup \{(1,1)(n,m),(1,m)(\delta,1)\}$, then $f_{\mathcal{B}_1}(e) = 0$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 0$ and $f_{\{C_i\}_{i=1}^5}(e) \le 2$. 6) If $e \in \{(1,j)(2,j+1) | j = 1,2,\dots,m-2\} \cup \{(\delta-1,j)(\delta,j+1) | j = 1,2,\dots,m-1\}$, then $f_{\mathcal{B}_1}(e) = 1$, $f_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K}}(e) = 1$ and $f_{\{C_i\}_{i=1}^5}(e) = 0$. 7) If

$e \in \{(1,j+1)(2,j),(\delta-1,j+1)(\delta,j)|j=1,2,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e)=2$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=1$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. 8) If $e=(1,m-1)(\delta,m)$ or $(1,m)(\delta,m-1)$, then $f_{\mathcal{B}_1}(e)=0$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=1$ and $f_{\{C_i\}_{i=1}^5}(e)=1$. 9) If $e\in\{\ (1,j)(\delta,j+1),(1,j+1)(\delta,j)|j=1,2,\ldots,m-2\}$, then $f_{\mathcal{B}_1}(e)=0$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)\leq 2$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. 10) If $e\in\{\ (\delta,j)(\delta+1,j+1),\{(\delta,j+1)(\delta+1,j)|j=1,2,\ldots,m-1\}$, then $f_{\mathcal{B}_1}(e)\leq 2$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=0$ and $f_{\{C_i\}_{i=1}^5}(e)\leq 1$. if e is not of the above form, then $f_{\mathcal{B}_1}(e)\leq 3$, $f_{\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}}(e)=0$ and $f_{\{C_i\}_{i=1}^5}(e)=0$. From all of the above, we have that $f_{\mathcal{B}(\theta_n\boxtimes C_m)}(e)\leq 3$.

Case 2. m and δ are even and n is odd. Then define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C^*,C_1,C_2,C_3,C_4,C_5\}$$

where

$$C^*=(1,1)(2,2)(3,1)(4,2)\ldots(n,1)(1,1)$$

and $\mathcal{B}_{a_ia_{i+1}},\mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1,\mathcal{A}_2,\mathcal{K},C_1,C_2$ and C_3 are as defined in Case 1 and

$$\begin{aligned} C_4 &= (1,1)(2,m)(3,1)(4,m)...(\delta,m)(1,1),\\ C_5 &= (1,m)(2,1)(3,m)(4,1)...(\delta,1)(1,m). \end{aligned}$$

By the same argument as in Case 1 of Theorem 2.6 of [11], we show that $\left(\bigcup_{i=1}^{n}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\{C^*\}$ is linearly independent. Following, more or less, the same proof of Case 1 by replacing C with C^*, we can show that $\mathcal{B}(\theta_n\boxtimes C_m)$ is a 4-fold basis for $\mathcal{C}(\theta_n\boxtimes C_m)$.

Case 3. m,n and δ are even. Then we define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C,C_1,C_2,C_3,C_4,C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}},\mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1,\mathcal{A}_2,\mathcal{K},C_1,C_2$, C_3,C_4 and C_5 are as defined in Case 2 and C is as in Case 1. By following, word by word, the proof of Case 2 after replacing C^* by C we get that $\mathcal{B}(\theta_n\boxtimes C_m)$ is a 4-fold basis.

Case 4. m is even and δ and n are odd. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 2.

Case 5. m is odd and n and δ are even. Then we define

$$\mathcal{B}(\theta_n\boxtimes C_m)=\left(\bigcup_{i=1}^{n-1}\mathcal{B}_{a_ia_{i+1}}\right)\cup\mathcal{B}^*_{a_na_1}\cup\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{K}\cup\{C\}\cup\{C_1,C_2,C_3,C_4,C_5\}$$

where $\mathcal{B}_{a_ia_{i+1}}$, $\mathcal{B}^*_{a_na_1}$ and C are as defined in Case 1. Also, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{K}$, C_2, C_4,C_5, are as in Case 3, and

$$\begin{aligned} C_1 &= (1,m)(2,m)(3,m)(4,m-1)\,(5,m)\,(6,m-1)\ldots(\delta,m-1)\,(1,m),\\ C_3 &= (1,m-1)(2,m)(3,m-1)(4,m)...(\delta,m)(1,m-1). \end{aligned}$$

Let $\mathcal{B}_1 = \bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}} \cup \mathcal{B}^*_{a_na_1} \cup \{C\}$. Note that $\mathcal{B}_1 =$B$(C_n \boxtimes C_m)$ is a basis for $\mathcal{C}(C_n \boxtimes C_m)$ (see Theorem 2.6 Case 2 of [11]). Thus, $\mathcal{B}_1$ is linearly independent. Note that $E(\mathcal{B}_1) \cap E(C_4) = \{(1,1)(2,m),(2,m)(3,1),\ldots,(\delta-1,1)(\delta,m)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_1 \cup \{C_4\}$ is linearly independent. Similarly, $E(\mathcal{B}_1 \cup \{C_4\}) \cap E(C_5) = \{(1,m)(2,1),(2,1)(3,m),\ldots,(\delta-1,m)(\delta,1)\}$ which is an edge set of a path. Thus, by Proposition 3.5, $\mathcal{B}_1 \cup \{C_4,C_5\}$ is linearly independent. Also, $E(\mathcal{B}_1 \cup \{C_4,C_5\}) \cap E(C_2) = \{(1,m)(1,1),(1,1)(\delta,m)\}$ which is an edge set of a forest. Thus, $\mathcal{B}_1 \cup \{C_2,C_4,C_5\}$ is linearly independent. Now, since $E(C_1) \cap E(C_3) = \varnothing$ and

$$\begin{aligned} E(C_1 \cup C_3) \cap E(\mathcal{B}_1 \cup \{C_2,C_4,C_5\}) &= \{(i,m-1)(i+1,m) | 1 \le i \le \delta-1\} \cup \\ \{(i,m)(i+1,m-1)|2 &\le i \le \delta-1\} \cup \{(1,m)(2,m),(2,m)(3,m)\}, \end{aligned}$$

which is an edge set of a tree, we have that $\mathcal{B}_1 \cup \{C_1,C_2,C_3,C_4,C_5\}$ is linearly independent. Now, by a similar argument as in Case 1, we can show that B$(\theta_n \boxtimes C_m)$ is a 4-fold basis.

Case 6. m and n are odd and δ is even. Then we define

$$\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_na_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C^*,C_1,C_2,C_3,C_4,C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}}, \mathcal{B}^*_{a_na_1}$, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{K}, C_1, C_2$, C_3, C_4 and C_5 are as defined in Case 5 and C^* is as in Case 2. To this end, we use the same argument as in Case 5 to show that $\mathcal{B}(\theta_n \boxtimes C_m)$ is a 4-fold basis.

Case 7. m,n and δ are odd. Then we define

$$\mathcal{B}(\theta_n \boxtimes C_m) = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_na_1} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{K} \cup \{C^*,C_1,C_2,C_3,C_4,C_5\},$$

where $\mathcal{B}_{a_ia_{i+1}}, \mathcal{B}^*_{a_na_1}$ are as defined above, $\mathcal{K}, \mathcal{A}_1$, $\mathcal{A}_2$ ad C_2 are as in Case 5 and C^* is as in Case 2. Also, we set,

$$\begin{aligned} C_1 &= (1,m-1)(2,m)(3,m-1)(4,m)...(\delta-1,m)(\delta,m)(1,m-1), \\ C_3 &= (1,m)(2,m-1)(3,m)(4,m-1)...(\delta-1,m-1)(\delta,m-1)(1,m), \\ C_4 &= (1,m)(2,m)(3,1)(4,m)(5,1)...(\delta,1)(1,m), \\ C_5 &= (1,1)(2,1)(3,m)(4,1)(5,m)...(\delta,m)(1,1). \end{aligned}$$

As in Case 2, we can see that $\mathcal{B}_1 = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{a_ia_{i+1}}\right) \cup \mathcal{B}^*_{a_n\mathcal{A}_1^{(i)}} \cup C^*$ is linearly independent. Now, the cycle C_1 contains the edge $(\delta,m)(1,m-1)$ which does not appear in any cycle of $\mathcal{B}_1$. Hence $\mathcal{B}_1 \cup \{C_1\}$ is linearly independent. Also, C_4 contains the edge $(\delta,1)(1,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1\}$. Thus, $\mathcal{B}_1 \cup \{C_1,C_4\}$ is linearly independent. C_5 contains the edge $(1,1)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1,C_4\}$. So, $\mathcal{B}_1 \cup \{C_1,C_4,C_5\}$ is linearly independent. C_2 contains the edge $(1,m)(\delta,m)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1,C_4,C_5\}$. Hence $\mathcal{B}_1 \cup \{C_1,C_2,C_4,C_5\}$ is linearly independent. Finally, C_3 contains the

edge $(1, m)(\delta, m-1)$ which does not appear in any cycle of $\mathcal{B}_1 \cup \{C_1, C_2, C_4, C_5\}$. Therefore, $\mathcal{B}_1 \cup \{C_1, C_2, C_3, C_4, C_5\}$ is linearly independent. To this end, to complete this case, we use the same argue as in Case 1.

Case 8. m and δ are odd and n is even. By relabeling the vertices of θ_n in the opposite direction, we get a similar case to Case 7. □

By noting that $C_m \boxtimes \theta_n$ is isomorphic to $\theta_n \boxtimes C_m$, we get the following result:

Corollary 3.1 *For any graph θ_n of order $n \geq 4$ and cycle C_m of order $m \geq 3$, we have $3 \leq b(C_m \boxtimes \theta_n) \leq 4$.*

References

[1] A.A. Ali, The basis number of the direct product of paths and cycles, *Ars Combin.* 27 (1988), 155-163.

[2] A.A. Ali, The basis number of complete multipartite graphs, *Ars Combin.* 28 (1989), 41-49.

[3] A.A. Ali and G.T. Marougi, The basis number of Cartesian product of some graphs, *The J. of the Indian Math. Soc.* 58 (1992), 123-134.

[4] A.S. Alsardary and J. Wojciechowski, The basis number of the powers of the complete graph, *Discrete Math.* 188 (1998), 13-25.

[5] M.K. Al-Qeyyam and M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs II, (To appear in JCMCC).

[6] J.A. Banks and E.F. Schmeichel, The basis number of n-cube, *J. Combin. Theory Ser. B* 33 (1982), no. 2, 95-100.

[7] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*, Wiley, New York, 2000.

[8] M.M.M. Jaradat, On the basis number of the direct product of graphs, *Australas. J. Combin.* 27 (2003), 293-306.

[9] M.M.M. Jaradat, The basis number of the direct product of a theta graph and a path, *Ars Combin.* 75 (2005), 105-111.

[10] M.M.M. Jaradat, An upper bound of the basis number of the strong product of graphs, *Discussion Mathematicea Graph Theory* 25 (2005), 391-406.

[11] M.M.M. Jaradat, The basis number of the strong product of trees and cycles with some graphs, *J. Combin. Math. Combin. Comput.* 58 (2006), 195–209.

[12] M.M.M. Jaradat, On the basis number and the minimum cycle bases of the wreath product of some graphs I, *Discussions Mathematicae Graph Theory* 26 (2006) 113-134.

[13] M.M.M. Jaradat, A new upper bound of the basis number of the lexicographic product of graphs, To appear in Ars Combin.

[14] M.M.M. Jaradat, An upper bound of the basis number of the semi-strong product of cycles with bipartite graphs, *Bulletin of the Korean Mathematical Society* 44 (2007) no. 3, 385-394.

[15] M. M. M. Jaradat, The basis number of the strong product of paths and cycles with bipartite graphs, *Missouri Journal of Mathematical Sciences* 19 (2007) no. 3, 219-230.

[16] M.M.M. Jaradat and M.K. Al-Qeyyam, On the basis number and the minimum cycle bases of the wreath product of wheels. *International Journal of Mathematical combinatorics*, Vol. 1 (2008), 52-62.

[17] M.M.M. Jaradat and M.Y. Alzoubi, Un upper bound of the basis number of the lexicographic products of graphs, *Australas. J. Combin.* 32 (2005), 305-312.

[18] M.M.M. Jaradat and MY. Alzoubi, On the basis number of the semi-strong product of bipartite graphs with cycles, *Kyungpook Math. J.* 45 (1) (2005), 45-53.

[19] M.M.M. Jaradat, M.Y. Alzoubi and E.A. Rawashdeh, The basis number of the Lexicographic product of different ladders*, SUT Journal of Mathematics* 40, no. 2, 91-101 (2004).

[20] S. MacLane, A combinatorial condition for planar graphs, *Fundamenta Math.* 28 (1937), 22-32.

[21] E.F. Schmeichel, The basis number of a graph, *J. Combin. Theory Ser. B*, 30 (1981), 123-129.

Math.Combin.Book.Ser. Vol.3 (2008), 51-55

Smarandache Curves in Minkowski Space-time

Melih Turgut and Süha Yilmaz

(Department of Mathematics of Buca Educational Faculty of Dokuz Eylül University, 35160 Buca-Izmir,Turkey.)

E-mail: melih.turgut@gmail.com, suha.yilmaz@yahoo.com

Abstract: A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve*. In this paper, we define a special case of such curves and call it *Smarandache* TB_2 *Curves* in the space E_1^4. Moreover, we compute formulas of its Frenet apparatus according to base curve via the method expressed in [3]. By this way, we obtain an another orthonormal frame of E_1^4.

Key Words: Minkowski space-time, Smarandache curves, Frenet apparatus of the curves.

AMS(2000): 53C50, 51B20.

§1. Introduction

In the case of a differentiable curve, at each point a tetrad of mutually orthogonal unit vectors (called tangent, normal, first binormal and second binormal) was defined and constructed, and the rates of change of these vectors along the curve define the curvatures of the curve in Minkowski space-time [1]. It is well-known that the set whose elements are frame vectors and curvatures of a curve, is called *Frenet Apparatus* of the curves.

The corresponding Frenet's equations for an arbitrary curve in the Minkowski space-time E_1^4 are given in [2]. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve*. We deal with a special Smarandache curves which is defined by the tangent and second binormal vector fields. We call such curves as *Smarandache* TB_2 *Curves*. Additionally, we compute formulas of this kind curves by the method expressed in [3]. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

§2. Preliminary notes

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^4 are briefly presented. A more complete elementary treatment can be found in the reference [1].

Minkowski space-time E_1^4 is an Euclidean space E^4 provided with the standard flat metric given by

[1]Received August 16, 2008. Accepted September 2, 2008.

$$g=-dx_1^2+dx_2^2+dx_3^2+dx_4^2,$$

where (x_1,x_2,x_3,x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v\in E_1^4$ can have one of the three causal characters; it can be space-like if $g(v,v)>0$ or $v=0$, time-like if $g(v,v)<0$ and null (light-like) if $g(v,v)$=0 and $v\neq 0$. Similary, an arbitrary curve $\alpha=\alpha(s)$ in E_1^4 can be locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector v is given by $\|v\|=\sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v)=\pm 1$. Next, vectors v,w in E_1^4 are said to be orthogonal if $g(v,w)=0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$.

Denote by $\{T(s),N(s),B_1(s),B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T,N,B_1,B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s),\alpha'(s))=\pm 1$.

Let $\alpha(s)$ be a curve in the space-time E_1^4, parametrized by arclength function s. Then for the unit speed space-like curve α with non-null frame vectors the following Frenet equations are given in [2]:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix}=\begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & \sigma & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where T,N,B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T)=g(N,N)=g(B_1,B_1)=1, g(B_2,B_2)=-1.$$

Here κ,τ and σ are, respectively, first, second and third curvature of the space-like curve α. In the same space, in [3] authors defined a vector product and gave a method to establish the Frenet frame for an arbitrary curve by following definition and theorem.

Definition 2.1 *Let $a=(a_1,a_2,a_3,a_4)$, $b=(b_1,b_2,b_3,b_4)$ and $c=(c_1,c_2,c_3,c_4)$ be vectors in E_1^4. The vector product in Minkowski space-time E_1^4 is defined by the determinant*

$$a\wedge b\wedge c=-\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2}$$

where e_1,e_2,e_3 and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1\wedge e_2\wedge e_3=e_4\ ,\ e_2\wedge e_3\wedge e_4=e_1\ ,\ e_3\wedge e_4\wedge e_1=e_2\ ,\ e_4\wedge e_1\wedge e_2=-e_3.$$

Theorem 2.2 *Let $\alpha = \alpha(t)$ be an arbitrary space-like curve in Minkowski space-time E_1^4 with above Frenet equations. The Frenet apparatus of α can be written as follows;*

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{3}$$

$$N = \frac{\|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha'}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}, \tag{4}$$

$$B_1 = \mu N \wedge T \wedge B_2, \tag{5}$$

$$B_2 = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{6}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|}{\|\alpha'\|^4} \tag{7}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}{\left\| \|\alpha'\|^2 .\alpha'' - g(\alpha', \alpha'').\alpha' \right\|} \tag{8}$$

and

$$\sigma = \frac{g(\alpha^{(IV)}, B_2)}{\|T \wedge N \wedge \alpha'''\| . \|\alpha'\|}, \tag{9}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of $[T, N, B_1, B_2]$ matrix.

§3. Smarandache Curves in Minkowski Space-time

Definition 3.1 *A regular curve in E_1^4, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Remark 3.2 Formulas of all Smarandache curves' Frenet apparatus can be determined by the expressed method.

Now, let us define a special form of Definition 3.1.

Definition 3.3 *Let $\xi = \xi(s)$ be an unit space-like curve with constant and nonzero curvatures κ, τ and σ; and $\{T, N, B_1, B_2\}$ be moving frame on it. Smarandache TB_2 curves are defined with*

$$X = X(s_X) = \frac{1}{\sqrt{\kappa^2(s) + \sigma^2(s)}} (T(s) + B_2(s)). \tag{10}$$

Theorem 3.4 *Let $\xi = \xi(s)$ be an unit speed space-like curve with constant and nonzero curvatures κ, τ and σ and $X = X(s_X)$ be a Smarandache TB_2 curve defined by frame vectors of $\xi = \xi(s)$. Then*

(i) The curve $X = X(s_X)$ is a space-like curve.

(ii) Frenet apparatus of $\{T_X, N_X, B_{1X}, B_{2X}, \kappa_X, \tau_X, \sigma_X\}$ Smarandache TB_2 curve $X = X(s_X)$ can be formed by Frenet apparatus $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$ of $\xi = \xi(s)$.

Proof Let $X = X(s_X)$ be a Smarandache TB_2 curve defined with above statement. Differentiating both sides of (10), we easily have

$$\frac{dX}{ds_X}\frac{ds_X}{ds} = \frac{1}{\sqrt{\kappa^2(s)+\sigma^2(s)}}(\kappa N + \sigma B_1). \tag{11}$$

The inner product $g(X', X')$ follows that

$$g(X', X') = 1, \tag{12}$$

where $'$ denotes derivative according to s. (12) implies that $X = X(s_X)$ is a space-like curve. Thus, the tangent vector is obtained as

$$T_X = \frac{1}{\sqrt{\kappa^2(s)+\sigma^2(s)}}(\kappa N + \sigma B_1). \tag{13}$$

Then considering Theorem 2.1, we calculate following derivatives according to s:

$$X'' = \frac{1}{\sqrt{\kappa^2+\sigma^2}}(-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma^2 B_2). \tag{14}$$

$$X^{'''} = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[\kappa\tau\sigma T + (-\kappa^3 - \kappa\tau^2)N + (\sigma^3 - \tau^2\sigma)B_1 + \kappa\tau\sigma B_2]. \tag{15}$$

$$X^{(IV)} = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[(...)T + (...)N + (...)B_1 + (\sigma^4 - \tau^2\sigma^2)B_2]. \tag{16}$$

Then, we form

$$\|X'\|^2 .X'' - g(X', X'').X' = \frac{1}{\sqrt{\kappa^2+\sigma^2}}[-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma B_2]. \tag{17}$$

Equation (17) yields the principal normal of X as

$$N_X = \frac{-\kappa^2 T - \tau\sigma N + \kappa\tau B_1 + \sigma B_2}{\sqrt{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}}. \tag{18}$$

Thereafter, by means of (17) and its norm, we write first curvature

$$\kappa_X = \sqrt{\frac{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}{\kappa^2+\sigma^2}}. \tag{19}$$

The vector product $T_X \wedge N_X \wedge X'''$ follows that

$$T_X \wedge N_X \wedge X''' = \frac{1}{A}\left\{\begin{array}{c} [\kappa\sigma(\kappa^2+\sigma^2)(\tau^2-\sigma)T + \kappa\tau\sigma^2(\kappa^2+\sigma)N \\ -\kappa^2\tau\sigma(\kappa^2+\sigma)B_1 + \kappa\tau(\kappa^2+\sigma^2)(\kappa^2+\tau^2)B_2] \end{array}\right\}, \tag{20}$$

where, $A = \frac{1}{\sqrt{(-\kappa^4+\tau^2\sigma^2+\kappa^2\tau^2+\sigma^2)(\kappa^2+\sigma^2)}}$. Shortly, let us denote $T_X \wedge N_X \wedge X'''$ with $l_1 T + l_2 N + l_3 B_1 + l_4 B_2$. And therefore, we have the second binormal vector of $X = X(s_X)$ as

$$B_{2X} = \mu\frac{l_1 T + l_2 N + l_3 B_1 + l_4 B_2}{\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}. \tag{21}$$

Thus, we easily have the second and third curvatures as follows:

$$\tau_X = \sqrt{\frac{(-l_1^2 + l_2^2 + l_3^2 + l_4^2)(\kappa^2 + \sigma^2)}{-\kappa^4 + \tau^2\sigma^2 + \kappa^2\tau^2 + \sigma^2}}, \tag{22}$$

$$\sigma_X = \frac{\sigma^2(\sigma^2 - \tau^2)}{(\kappa^2 + \sigma^2)\sqrt{-l_1^2 + l_2^2 + l_3^2 + l_4^2}}. \tag{23}$$

Finally, the vector product $N_X \wedge T_X \wedge B_{2X}$ gives us the first binormal vector

$$B_{1X} = \mu\frac{1}{L}\left\{\begin{array}{c} [(\kappa\sigma l_3 - \sigma^2 l_2 - \tau(\kappa^2 + \sigma^2)l_4]T - \sigma(\kappa^2 l_4 + \sigma l_1)N \\ +\kappa(\kappa^2 l_4 + \sigma l_1)B_1 + [\kappa^2(\sigma l_2 - \kappa^2 l_3) + \tau l_1(\kappa^2 + \sigma^2)]B_2 \end{array}\right\}, \tag{24}$$

where $L = \frac{1}{\sqrt{(-l_1^2+l_2^2+l_3^2+l_4^2)(\kappa^2+\sigma^2)(-\kappa^4+\tau^2\sigma^2+\kappa^2\tau^2+\sigma^2)}}$. $\square$

Thus, we compute Frenet apparatus of Smarandache TB_2 curves.

Corollary 3.1 *Suffice it to say that $\{T_X, N_X, B_{1X}, B_{2X}\}$ is an orthonormal frame of E_1^4.*

Acknowledgement

The first author would like to thank TUBITAK-BIDEB for their financial supports during his Ph.D. studies.

References

[1] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, 1983.

[2] J. Walrave, Curves and surfaces in Minkowski space. Dissertation, K. U. Leuven, Fac. of Science, Leuven, 1995.

[3] S. Yilmaz and M. Turgut, On the Differential Geometry of the curves in Minkowski space-time I, *Int. J. Contemp. Math. Sci.* 3(27), 1343-1349, 2008.

Math.Combin.Book.Ser. Vol.3 (2008), 56-78

The Characterization of Symmetric Primitive Matrices with exponent $n-3$

Lichao Huangfu[1] and Junliang Cai[2]

1. Beijing NO.4 High School, Beijing, P.R.China
2. School of Mathematical Sciences of Beijing Normal University, Beijing, 100875, P.R.China.

Email: caijunliang@bnu.edu.cn

Abstract: An $n \times n$ nonnegative matrix $A = (a_{ij})$ is said to be *Smarandachely primitive* if $A^k > 0$ for at least two integers $k > 0$ and *primitive* if for some integers $k > 0$. The least such integers k is called the *Smarandache exponent* or *exponent* of A and denoted by $\gamma^S(A)$ and $\gamma(A)$, respectively. The symmetric primitive matrices with exponent $\geq n-2$ has been described in articles [4]-[9]. In this paper the complete characterization of symmetric primitive matrices with exponent $n-3$ is obtained.

Key words: Smarandachely primitive matrix, Primitive matrix, Smarandache exponent, exponent, primitive graph.

AMS(2000): 05C10.

§1. Introduction

An $n \times n$ nonnegative matrix $A = (a_{ij})$ is said to be *Smarandachely primitive* if $A^k > 0$ for at least two integers $k > 0$ and *primitive* if for some integers $k > 0$. The least such integer k is called the *Smarandache exponent* or *exponent* of A and denoted by $\gamma^S(A)$ and $\gamma(A)$, respectively. The associated graph of *symmetric matrix* A, denoted by $G(A)$, is the graph with a vertex set $V(G(A)) = \{1, 2, \cdots, n\}$ such that there is an edge from i to j in $G(A)$ if and only if $a_{ij} > 0$. A graph G is called to be *primitive* if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$(not necessarily distinct), there is a walk from i to j with length k.The least such k is called the *exponent* of G, denoted by $\gamma(G)$. Clearly,a symmetric matrix A is primitive if and only if its associated graph $G(A)$ is primitive. And in this case, we have $\gamma(A) = \gamma(G(A))$. By this reason as above, we shall employ graph theory as a major tool and consider $\gamma(G(A))$ to prove our results.

Let SE_n be the exponent set of $n \times n$ symmetric primitive matrices. In 1986,Shao[4] proved $SE_n = \{1, 2, \cdots, 2n-2\} \setminus S$, where S is the set of all odd numbers among [n,2n-2] and gave the characterization of the matrix with exponent $2n-2$. In 1990,Wang[5] gave the characterization of the matrix with exponent $2n-4$. In 1991, Li[6] obtained the characterization with exponent

[1]Supported by Priority Discipline of Beijing Normal University and NNSFC (10271017).
[2]Received July 16, 2008. Accepted September 6, 2008.

$2n-6$. In 1995, Cai and Zhang[7] derived the complete characterization of symmetric primitive matrices with exponent $2n-2r(\geq n)$. In 2003, Cai and Wang[8] got the characterization with exponent $n-1$. In 2004,Cai[9] characterized the matrix with exponent $n-2$. The purpose of this paper is to go further into the problem and give the complete characterization of symmetric primitive matrices with exponent $n-3$.

§2. Some lemmas on $\gamma(G)$

For convenience, We will narrate the lemmas with graph theory below.

Lemma 2.1[4] *G is a primitive graph iff G is connected and has odd cycles.*

The *local exponent* from vertex u to v, denoted by $\gamma(u,v)$, is the least integer k such that there exists a walk of length l from u to v for all $l \geq k$.We denote $\gamma(u,u)$ by $\gamma(u)$ for short.

Lemma 2.2[4] *If G is a primitive graph, then*

$$\gamma(G) = \max_{u,v\in V(G)} \gamma(u,v).$$

We denote by $P(u,v)$ the shortest walk from u to v in G. The length of $P(u,v)$ is called the *distance* between u and v, denoted by $d_G(u,v)$. The *diameter* of G is defined as

$$diam(G) = \max_{u,v\in V(G)} d_G(u,v).$$

Let G_1 and G_2 be two subgraphs of $G.P(G_1,G_2)$ denotes the shortest walk between G_1 and G_2.Its length

$$d_G(G_1,G_2) = \min\{d_G(u,v) \mid u \in V(G_1), v \in V(G_2)\}.$$

Lemma 2.3[9] *Let G be a primitive graph,and let $u,v \in V(G)$. If there are two walks from u to v with length k_1 and k_2, respectively,where $k_1+k_2 \equiv 1(\mathrm{mod}\ )$,then*

$$\gamma(u,v) \leq \max\{k_1,k_2\} - 1.$$

Let $u,v \in V(G)$,we name the walk from u to v with different parity length to $d_G(u,v)$ a *dissimilar walk*, denoted by $W(u,v)$. The shortest (u,v)-dissimilar walk is called the *primitive walk* between u and v, denoted by $W_r(u,v)$, its length is denoted by $b(u,v)$ [9].

Lemma 2.4[8] *If G is a primitive graph, then*

$$\gamma(u,v) = b(u,v) - 1.$$

Therefore,

$$\gamma(G) = \max_{u,v\in V(G)} b(u,v) - 1.$$

Lemma 2.5[8] *Let G be a primitive graph, then*

(*i*) $\gamma(u,v) \geq d_G(u,v)$;

(*ii*) $\gamma(u,v) \equiv d_G(u,v) (\bmod 2)$;

(*iii*) $\gamma(G) \geq diam(G)$,and $\gamma(G) \equiv diam(G) (\bmod 2)$.

Lemma 2.6[8] *Suppose G is the primitive graph with order n. If there are $u, v \in V(G)$ such that $\gamma(u,v) = \gamma(G)$, then for any odd cycle C in G we have*

$$|V(P(u,v)) \cap V(C)| \leq n - \gamma(G).$$

Apparently, any (u,v)-dissimilar walk is inevitably correlative with some odd cycle. And for any odd cycle C, there is a (u,v)-dissimilar walk correlative with C, we denote it by $W(u,v,C)$. Therefore, there must be some smallest odd cycle C_0 such that $W_r(u,v) = W(u,v,C_0)$. We call C_0 a (u,v)-*primitive cycle* or the primitive cycle of $P(u,v)$. If there exists a (u,v)-shortest path which intersects with its primitive cycle C_0, then we can choose some (u,v)-shortest path, denoted by $P(u,v)$ might as well, such that their intersected vertexes can be arranged on a path.Set $p = |V(P(u,v)) \cap V(C_0)|$,then $p \leq \min\{n-\gamma(G), [\frac{n}{2}], \frac{1}{2}(|C_0|-1)\}$. Ulteriorly, we have

$$\begin{aligned} \gamma(u,v) &= \gamma(u,v,C_0) \\ &= d_G(u,C_0) + |P(C_0)| + d_G(v,C_0) - 1 \\ &= d_G(u,v) + |C_0| - 2(p-1) - 1, \end{aligned}$$

where $P(C_0)$ denotes the left part of C_0 which deletes the part in common with $P(u,v)$. If the (u,v)-shortest path has at most one intersected vertex with its primitive cycle C_0, there must be $w \in V(C_0)$ such that $d_G(u,C_0) = d_G(u,w)$ and $d_G(v,C_0) = d_G(v,w)$. Further we have

$$\begin{aligned} \gamma(u,v) &= \gamma(u,v,C_0) \\ &= d_G(u,C_0) + |C_0| + d_G(v,C_0) - 1 \\ &= d_G(u,w) + |C_0| + d_G(v,w) - 1. \end{aligned}$$

§3. Constructions of graphs

Let G be a primitive graph with order n. If there exists a vertex $w \in V(G)$ such that $\gamma(w) = \gamma(G)$, we call G a *graph of the first type*, otherwise a *graph of the second type*. Firstly, we define a class of graphs $\mathcal{N}_{n-3}$ as follows:

Denote the set $\mathcal{N}_{n-3} = \mathcal{N}_{n-3}^{(1)} \cup \mathcal{N}_{n-3}^{(3)} \cup \cdots \cup \mathcal{N}_{n-3}^{(n-2)}$,where $\mathcal{N}_{n-3}^{(d)} (1 \leq d \leq n-2, d \equiv 1 (\bmod 2), n \equiv 1 (\bmod 2))$ are defined as follows.

Let $n = 2r+3$ and $K = (V,E)$ be a graph, where the vertex set $V = \bigcup\limits_{0 \leq i \leq r} V_i$ with $V_i \cap V_j = \phi (0 \leq i < j \leq r)$ and $V_k = \{u_{l,k} \mid l = 1,2,\cdots,r+3\} (k = 0,1,\cdots,r)$,the edge set $E = E_1 \cup E_2$ with $E_1 = \{uv \mid u \in V_i, v \in V_{i+1}, 0 \leq i \leq r-1\}$ and $E_2 = \{uv \mid u,v \in V_r\}$. For any odd number d such that $1 \leq d \leq n-2$, let $t = r - \frac{1}{2}(d-1)$. We put the path $P_t = u_{1,0}u_{1,1}\cdots u_{1,t}$ and the cycle $C_d = u_{1,t}u_{1,t+1}\cdots u_{1,r}u_{2,r}\cdots u_{2,t+1}u_{1,t}$,and set $K_{(d)} = P_t \cup C_d$ which we call it a *structural graph*. Let the set of induced subgraphs with order n of K which contain $K_{(d)}$ be

$K^{(d)}$. For any $N \in K^{(d)}$, we denote the spanning subgraph of N which contains subgraph $K_{(d)}$ by $N_{(d)}$, and define the set of graphs $\mathcal{N}^{(d)}$ as:

$$\mathcal{N}^{(d)} = \{N_{(d)} \mid N \in K^{(d)}, 1 \leq d \leq n-2, d \equiv 1 (\mathrm{mod}\ 2)\}.$$

We mark the graphs of $\mathcal{N}^{(d)}$ with $\mathcal{N}^{(d)}_{n-3}$ which satisfy the following qualifications:

(1) $diam(N_{(d)}) \leq n-3$;

(2) For any odd number $d' > d$, there doesn't exist the graph $K_{(d')}$ in $N_{(d)}$;

(3) Let x be the vertex of $N_{(d)}$ such that $d_{N_{(d)}}(x, C_d) > t$, then there must exist a odd cycle C such that:

$$2d_{N_{(d)}}(x, C) + |C| \leq n-2.$$

Let $u_i \in V(P(x, C_d)) \cap P_t (i \leq t)$ be the vertex with the smallest subscript. If C is the odd cycle which doesn't intersect with $K_{(d)}$ and has at most one intersected vertex with $P(x, u_i)$(The shortest path from C to $P(x, u_i)$ is denoted by $P(w, z)$, where $w \in V(P(x, u_i))$ and $z \in V(C)$. And it suggests that C and $P(x, u_i)$ has only one vertex in common if $w = z$), and such that $2d_{N_{(d)}}(w, z) + |C|$ is as small as possible, then

(i) if $|C| + d = 4$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) + |C| = t+3$, then we must have

$$2d_{N_{(d)}}(w, z) + |C| \neq 2(t-i) + d.$$

(ii) if $|C| = d = 1$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) = t+1$, then we must have

$$d_{N_{(d)}}(w, z) \neq t - i.$$

(iii) if $|C| = d = 1$ and $d_{N_{(d)}}(x, u_i) + d_{N_{(d)}}(w, z) = t+2$, then we must have

$$|d_{N_{(d)}}(w, z) - (t-i)| \geq 6.$$

Another class of graphs $\mathcal{M}_{n-3}$ is defined as follows:

Let $n-3 = m+2r$, then $n-3 \equiv m (\mathrm{mod}\ 2)$. Let $T = (U, F)$ be a graph, where the vertex set $U = \bigcup\limits_{0 \leq i \leq r} U_i$ with $U_i \cap U_j = \phi (0 \leq i < j \leq r)$ and $U_i = \{u_{i,k} \mid k = 0, 1, \cdots, n-1\} (i = 0, 1, 2, \cdots, r)$, the edge set $F = F_1 \cup F_2 \cup F_3$ with $F_1 = \{u_{i,j} u_{k,l} \mid j+l+i+k \equiv 1 (\mathrm{mod}\ 2)\}$,$F_2 = \{uv \mid u, v \in U_r\}$ and $F_3 = \{uv \mid u \in U_{r-1}, v \in U_r\}$.We defined the set of graphs $\mathcal{M}_{n-3} = \mathcal{M}^{(0)}_{n-3} \cup \mathcal{M}^{(1)}_{n-3} \cup \mathcal{M}^{(2)}_{n-3} \cup \mathcal{M}^{(3)}_{n-3}$ as follows:

(i) Construction of $\mathcal{M}^{(0)}_{n-3}$: Let d_0, d_1 be the odd numbers such that $1 \leq d_0, d_1 \leq 5$ and $2 \leq d_0 + d_1 \leq 6$,and t_0, t_1 be the positive numbers such that $2r+1 = 2t_0 + d_0 \leq 2t_1 + d_1$ and $m + t_0 + t_1 + d_0 + d_1 \leq n+1$. We put the path $P_0 = u_{0,j} u_{1,j} \cdots u_{t_0,j}$ and the path $P_1 = u_{0,i} u_{1,i} \cdots u_{t_1,i} (0 \leq h \leq i < j \leq m+h \leq n-1)$. Let C_{d_0} be the cycle with length d_0 which has only one intersected vertex $u_{t_0,j}$ with P_0, while C_{d_1} be the cycle with length d_1 which has only one intersected vertex $u_{t_1,i}$ with P_1 and doesn't intersect with C_{d_0}. Put $K_{d_0,d_1} = P(u_{0,h}, u_{0,m+h}) \cup P_0 \cup P_1 \cup C_0 \cup C_1$, and call it a *structural graph*. Let $V(d_0, d_1) = V_1(d_0, d_1) \cup V_2(d_0, d_1)$, where $V_1(d_0, d_1) = V(K_{d_0,d_1})$ with $|V_1(d_0, d_1)| = m + t_0 + t_1 + d_0 + d_1 - 1 \leq n$,and $V_2(d_0, d_1) \subseteq U \setminus V_1(d_0, d_1)$ with $|V_2(d_0, d_1)| = t_0 + 3 - t_1 - d_1 \leq 2$. Therefore, we have

$|V(d_0,d_1)| = n$. We choose the connected subgraph T_{d_0,d_1} of $T[V(d_0,d_1)]$ to form the set of graphs $\mathcal{M}_{n-3}^{(0)}$, where T_{d_0,d_1} satisfies that:

(1) $diam(T_{d_0,d_1}) \leq n-3$;

(2) $V(T_{d_0,d_1}) = V(d_0,d_1)$, and $E(K_{d_0,d_1}) \subseteq E(T_{d_0,d_1})$;

(3) there doesn't exist a path P_2 and an cycle C_{d_2} such that $2t_2+d_2 < 2t_0+d_0$ and they have only one common vertex $u_{t_2,l}$,where $P_2 = u_{0,l}u_{1,l}\cdots u_{t_2,l}$ with length $t_2>0$ and C_{d_2} is an odd cycle with length d_2;

(4) if there exist a $(x_{a,j},y_{b,i})$-path with length $p = t_0+4-t_1-d_1 \leq 3$ which connects $P_0\cup C_0$ to $P_1\cup C_1$ in $T_{d_0,d_1}-E(K_{d_0,d_1})$, where $0\leq a\leq t_0$ and $0\leq b\leq t_1$, then we have

$$a+b+p > j-i, a+b+i+j \equiv p (\mathrm{mod}\ 2),$$

and

$$(2t_0+d_0)-(2t_1+d_1)-(p+i-j) \leq a-b \leq p+i-j;$$

(5) if there exists a vertex x in T_{d_0,d_1} such that $d_{T_{d_0,d_1}}(x,C_0) \geq t_0$ and $d_{T_{d_0,d_1}}(x,C_1) \geq t_0+\frac{1}{2}(d_0-d_1)$, there must exist an odd cycle C such that

$$2d_{T_{d_0,d_1}}(x,C)+|C| < m+2r+1.$$

(*ii*) Construction of $\mathcal{M}_{n-3}^{(1)}$: Let $m+2t_0+3=n, t_0\geq 0$. Let $C_{t_0} = u_{0,i}\cdots u_{t_0,i}u_{t_0,i+2}\cdots u_{1,i+2}u_{0,i} (0\leq h\leq i\leq m+h\leq n-1)$, then $|C_{t_0}| = 2t_0+1$(C_{t_0} is a loop on $u_{0,i}$ if $t_0=0$). Put the graph $K_{m,t_0} = P(u_{0,h},u_{0,m+h})\cup C_{t_0}$, and call it a *structural graph*. Let $V(m,t_0) = V_1(m,t_0)\cup V_2(m,t_0)$, where $V_1(m,t_0) = V(K_{m,t_0})$ and $V_2(m,t_0)\subseteq U\setminus V_1(m,t_0)$ with $|V_2(m,t_0)| = 2$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}_{n-3}^{(1)}$, where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \leq n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$, and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length $2t_0+1$ that has only one intersected vertex with $P(u_{0,h},u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h},u_{0,m+h}),C_d) > 0$;

(4) if there exists a $(u_{b,i},u_{a,i+2})$-path with length $p\leq 3$ which divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0\leq a,b\leq t_0$, then a,b must satisfy that:if $a+b\equiv p (\mathrm{mod}\ 2)$, then $|a-b|\leq p$; if $a+b+1\equiv p(\mathrm{mod}\ 2)$, then $a+b+p\geq 2t_0+1$;

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x,C_{t_0}) \geq \frac{1}{2}m$, there must be an odd cycle C such that

$$2d_{T_{m,t_0}}(x,C)+|C| < m+2r+1;$$

(*iii*) Construction of $\mathcal{M}_{n-3}^{(2)}$: Let $m+2t_0+3 = n, t_0\geq 0$. We put the cycle $C_{t_0} = u_{0,i}\cdots u_{t_0,i}zu_{t_0,i+1}\cdots u_{0,i+1}u_{0,i} (0\leq h\leq i< i+1\leq m+h\leq n-1)$, where $z = u_{t_0+1,i}$ or $u_{t_0+1,i+1}$,then $|C_{t_0}| = 2t_0+3$. Put $K_{m,t_0} = P(u_{0,h},u_{0,m+h})\cup C_{t_0}$, and we call it a *structural graph*. Let $V(m,t_0) = V_1(m,t_0)\cup V_2(m,t_0)$, where $V_1(m,t_0) = V(K_{m,t_0})$ and $V_2(m,t_0)\subseteq$

$U \setminus V_1(m,t_0)$ with $|V_2(m,t_0)|=1$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}^{(2)}_{n-3}$,where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \le n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$, and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length less than $2(t_0+q)-1$ which have $q(1 \le q \le 2)$ intersected vertexes with $P(u_{0,h}, u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h}, u_{0,m+h}), C_d) > 0$;

(4) if there exists a $(u_{b,i}, u_{a,i+1})$-path with length $p \le 2$ that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0 \le a, b \le t_0+1$, then a, b must satisfy that: if $a+b \equiv p(\text{mod } 2)$, then $a+b+p \ge 2t_0+2$; if $a+b+1 \equiv p(\text{mod } 2)$,then $|a-b| \le p+1$;

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x, C_{t_0}) \ge \frac{1}{2}m - 1$, there must be an odd cycle C such that

$$2d_{T_{m,t_0}}(x, C) + |C| < m + 2r + 1.$$

(iv) Construction of $\mathcal{M}^{(3)}_{n-3}$: Let $m + 2t_0 + 1 = n, t_0 \ge 0$. We put the cycle $C_{t_0} = u_{0,k-1} \cdots u_{t_0,k-1} u_{t_0,k+1} \cdots u_{0,k+1} u_{0,k} u_{0,k-1} (0 \le h \le k-1 < k+1 \le m+h \le n-1)$, then $|C_{t_0}| = 2t_0+3$. Put $K_{m,t_0} = P(u_{0,h}, u_{0,m+h}) \cup C_{t_0}$, and call it a *structural graph.* Put $V(m,t_0) = V(K_{m,t_0})$. We choose the connected subgraph T_{m,t_0} of $T[V(m,t_0)]$ to form the set of graphs $\mathcal{M}^{(3)}_{n-3}$, where T_{m,t_0} satisfies that:

(1) $diam(T_{m,t_0}) \le n-3$;

(2) $V(T_{m,t_0}) = V(m,t_0)$,and $E(K_{m,t_0}) \subseteq E(T_{m,t_0})$;

(3) neither does there exist an odd cycle with length less than $2(t_0+q)-3$ which have $q(1 \le q \le 3)$ intersected vertexes with $P(u_{0,h}, u_{0,m+h})$, nor does there exist an odd cycle C_d with length d such that $2t+d < 2t_0+1$ in T_{m,t_0}, where $t = d_{T_{m,t_0}}(P(u_{0,h}, u_{0,m+h}), C_d) > 0$;

(4) if there exist an edge $u_{b,k-1}u_{a,k+1}$ that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $0 \le a, b \le t_0$, then a, b must satisfy that:

$$a+b \equiv 1(\text{mod } 2), |a-b| \le 3;$$

if there exists an edge $v_k x_a$(or $v_k y_b$) that divides up C_{t_0} in $T_{m,t_0} - E(K_{m,t_0})$, where $1 \le a \le t_0$(or $1 \le b \le t_0$), then a(or b) must satisfy that: $a = 2$(or $b = 2$), or $a = 1$(or $b = 1$)(iff $t_0 = 1$);

(5) if there exists a vertex x in T_{m,t_0} such that $d_{T_{m,t_0}}(x, C_{t_0}) \ge \frac{1}{2}m - 2$, there must exist an odd cycle C such that

$$2d_{T_{m,t_0}}(x, C) + |C| < m + 2r + 1.$$

§4. Main results and proofs

Theorem 4.1 *G is a graph with order n of the first type with $\gamma(G) = n-3$ iff $G \in \mathcal{N}_{n-3}$.*

Proof For the necessity, suppose G is a graph with order n of the first type with $\gamma(G) = n-3$. Then there must be a vertex u_0 and an odd cycle C in G such that

$$\gamma(u_0) = \gamma(u_0, C) = \gamma(G) = n-3.$$

We choose u_0 and C such that $d = |C|$ is as great as possible, and denote $C = C_d$. Note that

$$\gamma(G) = \gamma(u_0) \equiv d_G(u_0, u_0)(\text{mod } 2), d_G(u_0, u_0) = 0,$$

we set $\gamma(G) = 2r$. So we get $n = 2r + 3$.

Let $t = d_G(u_0, C_d)$,then

$$\gamma(u_0) = 2t + d - 1 = 2r = n - 3.$$

Thus we get

$$n = 2t + d + 2, t = r - \frac{1}{2}(d - 1), 1 \leq d \leq 2r + 1.$$

We put the path $P_t = P(u_0, C_d) = u_0 u_1 \cdots u_t$, the cycle $C_d = u_t u_{t+1} \cdots u_{t+d-1} u_t$, and let

$$\begin{aligned} V_1(t,d) &= V(P_t \cup C_d), V_2(t,d) = V(G) \setminus V_1(t,d), \\ E_1(t,d) &= E(P_t \cup C_d), E_2(t,d) = E(G) \setminus E_1(t,d). \end{aligned}$$

Thus

$$n_1 = |V_1(t,d)| = t + d, n_2 = |V_2(t,d)| = t + 2.$$

It suggests above that there is a structural graph $K_{(d)} = P_t \cup C_d$ in G.To testify that $G \in \mathcal{N}_{n-3}^{(d)} \subset \mathcal{N}_{n-3}$,we shall prove that: (a) G meets the construct qualifications of $\mathcal{N}_{n-3}^{(d)}$, and (b) G is a subgraph of K.

(a) Note that $diam(G) \leq \gamma(G) = n - 3$, then the first construct qualification meets.By the choose of C_d, there doesn't exist the structural graph $K_{(d')}$(d' is an odd number with $d' > d$) in G, thus the second qualification meets. Suppose that there exists a vertex x such that $d_G(x, C_d) > t$, then

$$\gamma(x, C_d) = 2d_G(x, C_d) + d - 1 > 2t + d - 1 = n - 3.$$

If $2d_G(x, C) + |C| > n - 2$ for any odd cycle C which is different from C_d in G, we can get

$$\gamma(x, C) = 2d_G(x, C) + |C| - 1 > n - 3.$$

Thus we get a contradiction

$$\gamma(G) \geq \gamma(x) > n - 3 = \gamma(G).$$

Let $u_i \in V(P(x, C_d)) \cap P_t (i \leq t)$ be the vertex with the smallest subscript. Then $P(x, u_i)$ is a shortest path from C to P_t. Let C be the odd cycle which doesn't intersect with $K_{(d)}$ and has at most one intersected vertex with $P(x, u_i)$(The shortest path from C to $P(x, u_i)$ is denoted by $P(w, z)$, where $w \in V(P(x, u_i))$ and $z \in V(C)$. It suggests that C and $P(x, u_i)$ have only one vertex in common if $w = z$), and such that $2d_{N_{(d)}}(w, z) + |C|$ is as small as possible. Note that

$$\begin{aligned} \gamma(x, u_0, C) &\leq d_G(u, u_0) + 2d_G(w, z) + |C| - 1, \\ \gamma(x, u_0, C_d) &\leq d_G(u, u_0) + 2d_G(u_i, u_t) + d - 1, \end{aligned}$$

we then have

$$\begin{aligned}&\gamma(x,u_0,C)+\gamma(x,u_0,C_d)\\&=2(d_G(u,u_0)+d_G(u_i,u_t)+d_G(w,z)+|C|+d-1)-(d+|C|).\end{aligned}$$

(1) Suppose that $|C|+d=4$. If $d_G(x,u_i)+d_G(w,z)+|C|=t+3$ and $2d_G(w,z)+|C|=2(t-i)+d$, then we have

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+2=|V_2(d)|.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2n-4=2(n-2)$$

and

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)=n-2.$$

(2) Suppose that $|C|=d=1$. If $d_G(x,u_i)+d_G(w,z)=t+1$ and $d_G(w,z)=t-i$. Then we have

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+1=|V_2(d)|-1.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n-1.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2(n-1)-2=2(n-2)$$

and

$$\gamma(x,u_0,C)=\gamma(x,u_0,C_d)=n-2.$$

(3) Suppose that $|C|=d=1$.If $d_G(x,u_i)+d_G(w,z)=t+2$ and $|d_G(w,z)-t-i|<6$. Then we have

$$|\gamma(x,u_0,C)-\gamma(x,u_0,C_d)|<6,$$

and

$$d_G(x,u_i)+d_G(w,z)+|C|-1=t+2=|V_2(d)|.$$

Therefore,

$$d_G(x,u_0)+d_G(w,z)+d_G(u_i,u_t)+|C|+d-1=n.$$

Thus we get

$$\gamma(x,u_0,C)+\gamma(x,u_0,C_d)=2n-2=2(n-1).$$

Note that

$$\gamma(x, u_0, C) \equiv \gamma(x, u_0, C_d)(\text{mod } 2).$$

Hence we get

$$\min\{\gamma(x, u_0, C), \gamma(x, u_0, C_d)\} \geq n - 2.$$

The three cases lead to a common contradiction

$$\gamma(x, u_0) = \min\{\gamma(x, u_0, C), \gamma(x, u_0, C_d)\} \geq n - 2.$$

So the third qualification meets.

(b) Let

$$V(G) = U_0 \cup U_1 \cup \cdots \cup U_{r-1} \cup U_r,$$

where

$$\begin{aligned} U_i &= \{u \mid d_G(u_0, u) = i, u \in V(G)\}, \\ U_r &= \{u \mid d_G(u_0, u) \geq r, u \in V(G)\}, \end{aligned} \qquad (i = 0, 1, \cdots, r-1).$$

Then $G[U_i](i = 0, 1, \cdots, r-1)$ must be a null graph. Otherwise, there must be some odd cycle in $G' = G[U_0 \cup U_1 \cup \cdots \cup U_{r-1}]$. Let C be the odd cycle such that $d_G(u_0, C) + \frac{1}{2}(|C| - 1)$ is as small as possible in G'. Then we have

$$d_G(u_0, C) + \frac{1}{2}(|C| - 1) < r.$$

This implies a contradiction

$$\gamma(u_0) \leq \gamma(u_0, C) = 2d_G(u_0, C) + |C| - 1 < 2r = n - 3 = \gamma(u_0).$$

Note that $|U_i| \geq 1(i = 0, 1, \cdots, r)$. Then we have

$$|U_i| \leq 2r + 3 - r = r + 3.$$

So we can assert that G is a subgraph of K. Therefore, $G \in \mathcal{N}_{n-3}^{(d)} \subset \mathcal{N}_{n-3}$.

For the sufficiency, without loss of generality, we let $G \in \mathcal{N}_{n-3}^{(d)}$ with $1 \leq d \leq n-2$ and $d \equiv 1(\text{mod } 2)$. It is obvious that G is connected and has $K_{(d)} = P_t \cup C_d$ as its structural graph.

In the following argument, we shall prove two results:

(1) $\gamma(u_0) = n - 3$

Clearly, we have

$$\gamma(u_0, C_d) = 2d_G(u_0, C_d) + |C_d| - 1 = 2t + d - 1 = n - 3.$$

Hence we have $n = 2t + d + 2$. Put

$$n_1 = |V_1(d)| = |V(P_t \cup C_d)| = t + d,$$

and

$$n_2 = |V_2(d)| = |V(G) \setminus V_1| = t + 2.$$

If there is an odd cycle C in G such that $\gamma(u_0, C) < n-3 = 2r$, then $2d_G(u_0, C) + |C| - 1 < 2r$, i.e. $d_G(u_0, C) + \frac{1}{2}(|C| - 1) < r$. This implies that $G[U']$ contains the odd cycle C,where $U' = \{u \| \ d_G(u_0, u) < r, u \in V(G)\}$. Because the induced subgraph $K[V']$ of K about $V' = \{u \| \ d_K(u_0, u) < r, u \in V(K)\}$ is bipartite, its subgraph $G[U']$ doesn't contain any odd cycles,a contradiction. So we have $\gamma(u_0) = n-3$.

(2) $\forall\ u, v \in V(G), \gamma(u, v) \leq n-3$

It is obvious that $\gamma(u) \leq n-3$ for any vertex in G. In what follows, it suffices to prove $\gamma(u, v) \leq n-3$ for any two distinct vertexes u and v in $V(G)$.

If $d_G(u, C_d) + d_G(v, C_d) \leq 2t$, We can easily get $\gamma(u, v) \leq n-3$. So we put $d_G(u, C_d) + d_G(v, C_d) > 2t$, and without loss of generality we let $d_G(u, C_d) > t$,then there must be an odd cycle C in G such that $2d_G(u, C) + |C| \leq n-2$. Suppose that $V(P(u, C)) \cap V(P_t) \neq \phi$, let $w \in V(P(u, C)) \cap V(P_t)$ be the first vertex along $P(u, C)$ from u to C, then $d_G(u, w) > d_G(u_0, w)$. We then have

$$\begin{aligned}
\gamma(u_0) &\leq \gamma(u_0, C) \leq 2(d_G(u_0, w) + d_G(w, C)) + |C| - 1 \\
&< 2(d_G(u, w) + d_G(w, C)) + |C| - 1 \\
&= 2d_G(u, C) + |C| - 1 \leq n-3 = \gamma(u_0),
\end{aligned}$$

a contradiction. Therefore $P(u, C)$ doesn't intersect with P_t.

Let M be the component with u of $G[V_2(d)]$ in G, we shall complete our arguments in the following three cases:

(I) $V(C) \cap V(C_d) \neq \phi$

By the connectivity of G and $|V_2| = t+2$, we have $d_G(u, C_d) = t+1$ or $t+2$ which correspond to the following six cases.

(a) $d_G(u, C_d) = t+2, d_G(v, C_d) = t-1$

If $v \in V(P_t)$, we have

$$\begin{aligned}
\gamma(u, v) &\leq \gamma(u, v, C_d) \leq d_G(u, C_d) + d_G(v, C_d) + |C_d| - 2 \\
&= (t+2) + (t-1) + d - 2 = 2t + d - 1 = n-3.
\end{aligned}$$

If $v \in V(P(u, C))$, we have

$$\begin{aligned}
\gamma(u, v) &\leq \gamma(u, v, C) = d_G(u, C) + d_G(v, C) + |C| - 1 \\
&< 2d_G(u, C) + |C| - 1 \leq n-3.
\end{aligned}$$

(b) $d_G(u, C_d) = t+2, d_G(v, C_d) = t$

If $v \in V(P_t)$,note that $P(u, C)$ has no intersected vertex with P_t, we then have

$$|V(P(u, v) \cup V(C_d)| = 2t + d + 2 = n.$$

Hence the odd cycle C such that $2d_G(u, C) + |C| \leq n-2$ must be a loop on $P(u, v)$, this means $|C| = 1$. So we get

$$\begin{aligned}
\gamma(u, v) &\leq \gamma(u, v, C) = d_G(u, v) + |C| - 1 \\
&= d_G(u, v) \leq diam(G) \leq \gamma(G).
\end{aligned}$$

If $v \in V(P(u,C))$, we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) = d_G(u,C) + d_G(v,C) + |C| - 1 \\ &< 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(c) $d_G(u, C_d) = t+2, d_G(v, C_d) = t+1$

This suggests that $v \in V(P(u,C))$, i.e. $uv \in E(P(u,C))$, hence we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) = d_G(u,C) + d_G(v,C) + |C| - 1 \\ &< 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(d) $d_G(u, C_d) = t+1, d_G(v, C_d) = t$

The argument is similar to (a).

(e) $d_G(u, C_d) = t+1, d_G(v, C_d) = t+1$

Let $uw \in E(P(u,C))$,there must be $vw \in E(G) \setminus (E(K_d) \cup E(P(u,C)))$. Hence we have

$$\begin{aligned}\gamma(u,v) &\leq \gamma(u,v,C) \leq d_G(u,C) + d_G(v,C) + |C| - 1 \\ &= 2d_G(u,C) + |C| - 1 \leq n-3.\end{aligned}$$

(f) $d_G(u, C_d) = t+1, d_G(v, C_d) = t+2$

The argument is similar to (c).

(II) $V(C) \cap V(C_d) = \phi, V(C) \cap V(P_t) \neq \phi$

Let $u_i, u_j \in V(C) \cap V(P_t)$ be the vertexes with the smallest and biggest subscripts respectively, where $i \leq j \leq t-1$. By the construct qualification (2), we have

$$2d(u_0, u_i) + |C| > 2d(u_0, u_t) + d,$$

i.e.

$$\frac{1}{2}(|C| - 1) \geq t - i + \frac{1}{2}(d+1).$$

By $d(u, C_d) \geq t+1$, we have

$$d(u,C) + \frac{1}{2}(|C| - 1) + (t-j) \geq t+1,$$

i.e.

$$d(u,C) + \frac{1}{2}(|C| - 1) \geq j+1.$$

Hence,

$$d(u,c) + |C| - (j-i+1) \geq t+1+\frac{1}{2}(d+1).$$

In addition, notice that $|V_2(d)| = t+2$. We have

$$d(u,C) + |C| - (j-i+1) \leq t+2.$$

So we have

$$t+1+\frac{1}{2}(d+1) \leq d(u,c) + |C| - (j-i+1) \leq t+2.$$

This means
$$d = 1, |C| = 2t - 2i + 3,$$
and
$$d(u, C) = i + j - t(i + j \geq t).$$

If $v \in V(M)$, it is obvious that
$$\gamma(u, v) < \gamma(u, C) \leq \gamma(G).$$

If $v \notin V(M)$, clearly we have
$$\begin{aligned}\gamma(u, v) &\leq \gamma(u, u_0) \leq \gamma(u, u_0, C) \leq d(u, C) + d(u_0, C) + \frac{1}{2}(|C| - 1)\\ &= (i + j - t) + i + (t - i + 1) = i + j + 1 < \gamma(G).\end{aligned}$$

(III) $V(C) \cap V(C_d) = \phi, V(C) \cap V(P_t) = \phi$

Let $u_i \in V(P(u, C_d)) \cap V(P_t)(i \leq t)$ be the vertex with the smallest subscript, then $P(u, u_i)$ is the shortest path from u to P_t. We shall discuss in the two following cases.

(a) Suppose C and $P(u, u_i)$ have at least two intersected vertexes. Then $|C| \geq 3$.

Let $v \in V(M)$.If $P(u, C)$ intersects with $P(v, C)$, then we have
$$\begin{aligned}\gamma(u, v) &\leq \gamma(u, v, C)\\ &\leq 2\max\{d(u, C), d(v, C)\} + |C| - 1\\ &\leq 2(|V_2(d)| - |C|) + |C| - 1\\ &= 2t - |C| + 3 \leq 2t \leq \gamma(G).\end{aligned}$$

If $P(u, C)$ doesn't intersect with $P(v, C)$, then we have
$$\gamma(u, v) \leq \gamma(u, v, C) \leq |V_2(d)| - 1 = t + 1 \leq \gamma(G).$$

Let $v \notin V(M)$ and $|V_1^{'}| = |V_1(d) \setminus V(P(u_0, u_i))| \geq 2$. Then we have
$$\gamma(u, v) \leq \gamma(u, v, C) \leq n - |V_1^{'}| - 1 \leq n - 3 = \gamma(G).$$

If $|V_1^{'}| = 1$, it means that $i = t - 1$ and $d = 1$. Note that $d(u, C_d) \geq t + 1$, we have $d(u, u_i) \geq i + 1 = t$. Note that $|V_2(d)| = t + 2, |C| \geq 3$,we have $|C| \leq 5$: if $|C| = 3$, there must be only two intersected vertexes of C and $P(u, u_i)$; if $|C| = 5$, there must be just three intersected vertexes of C and $P(u, u_i)$. Thus we can easily have
$$\gamma(u, v) \leq \gamma(u, u_0, C) \leq 2t \leq \gamma(G).$$

(b) Suppose that there is at most one intersected vertex of C and $P(u, u_i)$. Let $P(w, z)$ be the shortest path from C to $P(u, u_i)$, where $w \in V(P(u, u_i))$ and $z \in V(C)(w = z$ suggests that there is only one intersected vertex of C and $P(u, u_i))$.

Let $v \in V(M)$. If $P(u, C)$ doesn't intersect with $P(v, C)$, we have
$$\gamma(u, v) \leq \gamma(u, v, C) \leq |V_2(d)| - 1 = t + 1 \leq \gamma(G).$$

If $P(u,C)$ intersects with $P(v,C)$, note that $2d(u,C)+|C|\le 2t+d$, we then have

$$d(u,C)\le t+\frac{1}{2}(d-|C|).$$

If $d(v,C)<t+2-|C|$, i.e. $d(v,C)+|C|-1\le t$, we have

$$\begin{aligned}\gamma(u,v) &\le \gamma(u,v,C)\le d(u,C)+d(v,C)+|C|-1\\ &\le (t+\frac{1}{2}(d-|C|))+t\le 2t+d-1=\gamma(G).\end{aligned}$$

If $d(v,C)\ge t+2-|C|$, note that $d(v,C)+|C|\le |V_2(d)|=t+2$, we then have

$$d(v,C)=t+2-|C|.$$

Now it is clear that u is just on $P(v,C)$ and $d(v,C_d)\ge t+1$. So there must be an odd cycle C' such that

$$2d(v,C')+|C'|\le 2t+d.$$

If C' is a loop on $P(u,v)$, we then have

$$\gamma(u,v)\le d(u,v)\le diam(G)\le \gamma(G).$$

Otherwise, C' doesn't intersect with $P(u,v)$. This suggests that $d(u,C')\le d(v,C')$. Hence we have

$$\gamma(u,v)\le \gamma(v,C')\le \gamma(G).$$

If $|C'|\ge 3$,then C' must intersects with C. Similarly, $d(u,C')\le d(v,C')$. So we have

$$\gamma(u,v)\le \gamma(v,C')\le \gamma(G).$$

Let $v\notin V(M)$. Note that

$$\begin{aligned}&\gamma(u,u_0,C)=d(u,u_0)+2d(w,z)+|C|-1,\\ &\gamma(u,u_0,C_d)=d(u,u_0)+2d(u_i,u_t)+d-1,\end{aligned}$$

we have

$$\begin{aligned}\gamma(u,u_0,C) &+ \gamma(u,u_0,C_d)\\ &= 2(d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1)-(d+|C|).\end{aligned}$$

If $d+|C|\ge 6$, we have

$$\gamma(u,u_0)=\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Therefore, we get

$$\gamma(u,v)\le \gamma(u,u_0)\le \gamma(G).$$

In what follows, it suffices to discuss the case such that $|C|+d\le 4$.

Suppose that $|C|+d=4$ and $d(u,u_i)+d(w,z)+|C|\le t+2$, we have

$$d(u,u_i)+d(w,z)+|C|-1\le t+1=|V_2(d)|-1,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1\le n-1.$$

Hence we have

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2(n-1)-4=2(n-3).$$

This suggests that

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $d(u,u_i)+d(w,z)+|C|\ge t+3$, note that

$$d(u,u_i)+d(w,z)+|C|-1\le |V_2(d)|=t+2,$$

we then have

$$d(u,u_i)+d(w,z)+|C|-1=|V_2(d)|,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1=n.$$

Hence

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2n-4=2(n-2).$$

By the construction of the G, we have

$$2d(w,z)+|C|\ne 2(t-i)+d,$$

i.e.

$$\gamma(u,u_0,C)\ne\gamma(u,u_0,C_d).$$

This suggests that

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $|C|=d=1$ and $d(u,u_i)+d(w,z)\le t$, we then have

$$d(u,u_i)+d(w,z)+|C|-1=t=|V_2(d)|-2,$$

i.e.

$$d(u,u_0)+d(u_i,u_t)+d(w,z)+|C|+d-1\le n-2.$$

We then have

$$\gamma(u,u_0,C)+\gamma(u,u_0,C_d)\le 2(n-2)-2=2(n-3).$$

Thus we have

$$\min\{\gamma(u,u_0,C),\gamma(u,u_0,C_d)\}\le n-3.$$

Suppose that $d(u,u_i)+d(w,z)\ge t+1$. Note that

$$d(u,u_i)+d(w,z)+|C|-1\le |V_2(d)|=t+2,$$

we then have

$$t + 1 \le d(u, u_i) + d(w, z) \le t + 2.$$

If $d(u, u_i) + d(w, z) = t + 1$, we thus get

$$d(u, u_0) + d(u_i, u_t) + d(w, z) + |C| + d - 1 = n - 1.$$

It means that

$$\gamma(u, u_0, C) + \gamma(u, u_0, C_d) = 2(n - 1) - 2 = 2(n - 2).$$

Note that $d(w, z) \ne t - i$, we have

$$\gamma(u, u_0, C) \ne \gamma(u, u_0, C_d).$$

We therefore get

$$\min\{\gamma(u, u_0, C), \gamma(u, u_0, C_d)\} \le n - 3.$$

Suppose that $d(u, u_i) + d(w, z) = t + 2$, then we have

$$d(u, u_0) + d(u_i, u_t) + d(w, z) + |C| + d - 1 = n.$$

Hence

$$\gamma(u, u_0, C) + \gamma(u, u_0, C_d) = 2n.$$

If $|d(w, z) - t - i| > 6$, we then get

$$|\gamma(u, u_0, C) - \gamma(u, u_0, C_d)| > 6.$$

This suggests that

$$\min\{\gamma(u, u_0, C), \gamma(u, u_0, C_d)\} \le n - 3.$$

From those as above, we can easily get

$$\gamma(u, v) \le \gamma(u, u_0) \le \gamma(G).$$

Hence, $\forall\ u, v \in V(G)$, we have $\gamma(u, v) \le n - 3$. $\square$

Theorem 4.2 *G is a graph with order n of the second type with $\gamma(G) = n - 3$ iff $G \in \mathcal{M}_{n-3}$.*

Proof For the sufficiency, $\forall\ G \in \mathcal{M}_{n-3}$, we have $\gamma(G) = n - 3$ and $\gamma(w) < \gamma(G)$ for all $w \in V(G)$ by a direct verification.

Now for the necessity, suppose G is a graph of order n of the second type with $\gamma(G) = n-3$. Then there must be two distinct vertexes u and v and an odd cycle C_0 such that

$$\gamma(u, v) = \gamma(u, v, C_0) = \gamma(G) = n - 3.$$

We put the path $P(u, v) = v_0v_1 \cdots v_m$, where $v_0 = u$ and $v_m = v$ with $n - 3 \equiv m(\text{mod } 2)$. Without loss of generality,we set

$$n - 3 = m + 2r, d_0 = |C_0| \equiv 1(\text{mod } 2).$$

Suppose that C is an odd cycle in G, then we have

$$|V(P(u,v)) \cap V(C)| \leq n - \gamma(G) = 3.$$

In the following, we shall complete our arguments in four cases.

(I) Suppose that any odd cycle doesn't intersect with any (u,v)-shortest path in G, then we have

$$t_0 = d_G(P(u,v), C_0) > 0.$$

By the equation $\gamma(u,v) = \gamma(u,v,C_0)$, we can easily get

$$n = m + 2t_0 + d_0 + 2.$$

We put the path $P_0 = P(P(u,v),C_0) = x_0x_1\cdots x_{t_0}$, where $x_0 = v_j$ and $x_{t_0} \in V(C_0)$. Set

$$V_1 = V(P(u,v)) \cup V(C_0) \cup V(P_0), V_2 = V(G) \setminus V_1,$$

then we have

$$|V_1| = m + t_0 + d_0, |V_2| = t_0 + 2.$$

Suppose that the odd cycles C_1 and C_2 satisfy the following qualifications respectively.

$$\gamma(u) = \gamma(u, C_1), \gamma(v) = \gamma(v, C_2).$$

If $V(P(u,C_1)) \cap V(P_0) \neq \phi$ and $V(P(v,C_2)) \cap V(P_0) \neq \phi$, it is clear that

$$\gamma(u, C_1) = \gamma(u, C_0), \gamma(v, C_2) = \gamma(v, C_0).$$

Hence

$$\begin{aligned}
\gamma(u) &= \gamma(u,C_0) = 2d_G(u,C_0) + d_0 - 1 = 2d_G(u,x_{t_0}) + d_0 - 1 < n-3, \\
\gamma(v) &= \gamma(v,C_0) = 2d_G(v,C_0) + d_0 - 1 = 2d_G(v,x_{t_0}) + d_0 - 1 < n-3.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\gamma(G) = \gamma(u,v) &= \gamma(u,v,C_0) = d_G(u,x_{t_0}) + d_G(v,x_{t_0}) + d_0 - 1 \\
&< n - 3 = \gamma(G),
\end{aligned}$$

a contradiction. So we assume $V(P(u,C_1)) \cap V(P_0) = \phi$ without loss of generality. Suppose $v_i \in V(P(u,C_1)) \cap V(P(u,v))$ is the intersected vertex with the biggest subscript, put the path $P_1 = P(P(u,v),C_1) = y_0y_1\cdots y_{t_1}$ with $d_1 = |C_1|$ and $t_1 = d_G(v_i, C_1)$, where $y_0 = v_i$ and $y_{t_1} \in V(C_1)$. Then we have $V(P_0) \cap V(P_1) = \phi(i<j)$ and

$$t_1 \leq t_1 + d_1 - 1 \leq |V_2| \leq t_0 + 2.$$

By the choose of $P(u,v)$ and C_0, we have

$$2t_0 + d_0 \leq 2t_1 + d_1.$$

Hence

$$2t_1 + 2d_1 - 6 + d_0 \le 2t_0 + d_0 \le 2t_1 + d_1 \le 2t_0 + 5.$$

So we get

$$2 \le d_0 + d_1 \le 6, |t_0 - t_1| \le 2.$$

Set $K_{d_0,d_1} = P(u,v) \cup P_0 \cup P_1 \cup C_0 \cup C_1$, then we have

$$|V(K_{d_0,d_1})| = m + t_0 + t_1 + d_0 + d_1 - 1 \le n,$$

and

$$|V(G) \setminus V(K_{d_0,d_1})| = t_0 + 3 - t_1 - d_1 \le 2.$$

It is easy to verify that G meets the first three construct qualifications (1),(2) and (3) of T_{d_0,d_1}. We shall prove that G meets the other qualifications:

Suppose there exists a (x_a, y_b)-path with length p which connects $P_0 \cup C_0$ to $P_1 \cup C_1$ in $G - E(K_{d_0,d_1})$, where $0 \le a \le t_0$ and $0 \le b \le t_1$. Clearly,$p = t_0 + 4 - t_1 - d_1 \le 3$. Because any odd cycle doesn't intersect with the (u,v)-shortest path $P(u,v)$, then we have

$$a + b + p > j - i, a + b + i + j \equiv p (\mathrm{mod}\ 2),$$

and

$$\begin{aligned} d_G(v_0, v_i) + d_G(v_i, y_b) + p &+ d_G(x_a, v_j) + d_G(v_j, v_m) + 2d_G(x_a, x_{t_0}) + |C_0| - 1 \\ &\ge \gamma(u,v), \end{aligned}$$

and

$$\begin{aligned} d_G(v_0, v_i) + d_G(v_i, y_b) + p &+ d_G(x_a, v_j) + d_G(v_j, v_m) + 2d_G(y_b, y_{t_1}) + |C_1| - 1 \\ &\ge \gamma(u,v). \end{aligned}$$

Hence we have

$$i + b + p + a + m - j + 2(t_0 - a) + d_0 - 1 \ge m + 2t_0 + d_0 - 1,$$

and

$$i + b + p + a + m - j + 2(t_1 - b) + d_1 - 1 \ge m + 2t_0 + d_0 - 1.$$

Therefore,we get

$$(2t_0 + d_0) - (2t_1 + d_1) - (p + i - j) \le a - b \le p + i - j.$$

The qualification (4) thus meets.

Suppose that there exists a vertex x in G such that $d_G(x, C_0) \ge t_0$ and $d_G(x, C_1) \ge t_0 + \frac{1}{2}(d_0 - d_1)$, then we can conclude that

$$\gamma(x, C_0) \le \gamma(G), \gamma(x, C_1) \le \gamma(G).$$

Hence there must be an odd cycle C in G such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x,C)+|C|-1<n-3=m+2r.$$

The fifth qualification (5) meets.

Clearly, $G\subseteq T$, then we have $G\in\mathcal{M}_{n-3}^{(0)}$.

(II) Suppose that there exists an odd cycle C and a path $P(u,v)$ in G such that $|V(P(u,v))\cap V(C)|=1$, but for any odd cycle C' and any (u,v)-shortest path $P'(u,v)$, $|V(P'(u,v))\cap V(C')|\geq 2$ doesn't come into existence. Without loss of generality, we assume that C is the smallest odd cycle which meets the above qualifications, and $V(P(u,v))\cap V(C)=\{v_i\}$. Hence

$$n-3=\gamma(u,v)\leq\gamma(u,v,C)=m+|C|-1=|V(P(u,v)\cup C)|-1\leq n-1.$$

Note that $n-3\equiv m(\text{mod } 2)$, we have $m+|C|=n-2$ or $m+|C|=n$. Suppose that $m+|C|=n$, then we can assert that C isn't a primitive cycle of $P(u,v)$, and $V(G)=V(P(u,v))\cup V(C)$. Note that

$$m=d_G(u,v)\leq\gamma(u,v)=n-3,$$

then we have $|C|\geq 3$. Suppose that the odd cycles C_1 and C_2 satisfy the following equations respectively:

$$\gamma(u)=\gamma(u,C_1),\gamma(v)=\gamma(v,C_2).$$

If both C_1 and C_2 intersect with C, clearly we have

$$\gamma(u,C_1)=\gamma(u,C),\gamma(v,C_2)=\gamma(u,C).$$

It is easy to verify

$$\max\{\gamma(u,C),\gamma(v,C)\}\geq\gamma(G),$$

a contradiction. Hence, we assume that C_1 doesn't intersect with C without loss of generality. Therefore, C_1 must intersect with $P(u,v)$, and C_1 is a loop on v_i of $P(u,v)$(without loss of generality,we set $i<j$). This contradicts the choose of C.Hence,$m+|C|\neq n$. Therefore,we set $m+|C|=n-2$. We then have

$$n-3=\gamma(u,v)=\gamma(u,v,C).$$

We might as well put the cycle $C=C_0=y_0\cdots y_{t_0}x_{t_0}\cdots x_1y_0$, where $y_0=v_i$. Thus, we have

$$|C|=2t_0+1,n=m+2t_0+3.$$

We put the graph $K_{m,t_0}=P(u,v)\cup C_0$. It is obvious that its order is $n-2$. It is easy to verify that G meets the first three construct qualifications (1), (2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there is a (x_a,y_b)-path with length p which divides up C_0 in $G-E(K_{m,t_0})$, where $0\leq a,b\leq t_0$.Clearly,$p\leq 3$. Note that any odd cycle in G has only one intersected vertex with the (u,v)-shortest path, and C_0 is the smallest odd cycle which has only one intersected vertex with $P(u,v)$:

If $a+b+1 \equiv p(\text{mod } 2)$, then we have

$$d_G(v_i, x_a) + d_G(v_i, y_b) + p \geq |C_0|,$$

i.e.

$$a + b + p \geq 2t_0 + 1.$$

If $a+b \equiv p(\text{mod } 2)$, then we have

$$d_G(v_0, v_m) + 2d_G(x_a, v_i) + |C_0| - d_G(x_a, v_i) - d_G(v_i, y_b) + p - 1 \geq \gamma(u, v)$$

and

$$d_G(v_0, v_m) + 2d_G(v_i, y_b) + |C_0| - d_G(x_a, v_i) - d_G(v_i, y_b) + p - 1 \geq \gamma(u, v),$$

i.e.

$$m + a + 2t_0 - b + p \geq m + 2t_0$$

and

$$m + b + 2t_0 - a + p \geq m + 2t_0.$$

Hence, $|a-b| \leq p$, and the fourth qualification (4) comes into existence.

Suppose there exists a vertex x in G such that $d_G(x, C_0) \geq \frac{1}{2}m$, then we have $\gamma(x, C_0) \geq \gamma(G)$. Therefore, there must be an odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

The fifth qualification (5) comes into existence.

Clearly, $G \subseteq T$, then we have $G \in \mathcal{M}_{n-3}^{(1)}$.

(III) Suppose that there exists an odd cycle C and a path $P(u, v)$ in G such that $|V(P(u, v)) \cap V(C)| = 2$, but for any odd cycle C' and any (u, v)-shortest path $P'(u, v)$, $|V(P'(u, v)) \cap V(C')| \geq 3$ doesn't come into existence. Without loss of generality, we assume that C is the smallest odd cycle satisfying the above qualifications, and $V(P(u, v)) \cap V(C) = \{v_i, v_j (i < j)\}$. Clearly, $j = i + 1$. Hence, we have

$$\begin{aligned} n - 3 &= \gamma(u, v) \leq \gamma(u, v, C) = i + (m - j) + |C| - 2 \\ &= m + |C| - 3 = |V(P(u, v) \cup C)| - 2 \leq n - 2. \end{aligned}$$

Note that $n - 3 \equiv m(\text{mod } 2)$. We have $\gamma(u, v) = \gamma(u, v, C)$. Hence, C is a (u, v)-primitive cycle, where $n = m + |C|$. We might as well put the cycle $C = C_0 = y_0 \cdots y_{t_0} z x_{t_0} \cdots x_0 y_0$,where $y_0 = v_i$ and $x_0 = v_{i+1}$. Hence, we have

$$|C_0| = 2t_0 + 3, n = m + 2t_0 + 3.$$

We put the graph $K_{m,t_0} = P(u, v) \cup C_0$. It is obvious that its order is $n - 1$. It is easy to verify that G meets the first three construct qualifications (1),(2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there exists a (x_a, y_b)-path with length p which divides up C_0 in $G-E(K_{m,t_0})$, where $0 \le a, b \le t_0$.Clearly,$p \le 2$. Note that any odd cycle has at most two intersected vertexes with any (u,v)-shortest path in G, and C_0 is the smallest odd cycle which has just two intersected vertexes with $P(u,v)$.

(a) If $a+b+1 \equiv p(\text{mod } 2)$, then we have

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(v_j, x_a) + |C_0| &- d_G(v_j, x_a) - d_G(v_i, y_b) - d_G(v_i, v_j) + p - 1 \\ &\ge \gamma(u,v) \end{aligned}$$

and

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(v_i, y_b) + |C_0| &- d_G(v_j, x_a) - d_G(v_i, y_b) - d_G(v_i, v_j) + p - 1 \\ &\ge \gamma(u,v). \end{aligned}$$

Hence, we have

$$m + a + 2t_0 - b + p + 1 \ge m + 2t_0$$

and

$$m + b + 2t_0 - a + p + 1 \ge m + 2t_0.$$

Therefore, we have

$$|a - b| \le p + 1.$$

(b) If $a + b \equiv p(\text{mod } 2)$, because C_0 is the the smallest odd cycle which has just two intersected vertexes with $P(u,v)$, we have

$$d_G(v_j, x_a) + d_G(v_i, y_b) + d_G(v_i, v_j) + p \ge |C_0|.$$

We thus get

$$a + b + p \ge 2t_0 + 2.$$

Suppose that there exists a (z, x_a)-path(or (z, y_b)-path) with length p in $G - E(K_{m,t_0})$ which divides up C_0, where $0 \le a, b \le t_0$. Clearly, $p \le 2$. If $a + t_0 + 1 \equiv p(\text{mod } 2)$, note that C_0 is the smallest odd cycle which has just two intersected vertexes with $P(u,v)$, then we have

$$d_G(v_j, x_a) + p + d_G(z, y_{t_0}) + d_G(y_{t_0}, v_i) + d_G(v_i, v_j) \ge |C_0|.$$

We thus get

$$a + p \ge t_0 + 1.$$

If $a + t_0 \equiv p(\text{mod } 2)$, then we have

$$d_G(v_0, v_m) + 2d_G(x_a, v_j) + (p + d_G(x_a, x_{t_0}) + d_G(z, x_{t_0})) - 1 \ge \gamma(u,v),$$

i.e.

$$m + 2a + (p + t_0 - a + 1) - 1 \ge m + 2t_0.$$

We then get

$$a + p \ge t_0.$$

Using analogous argument, we can get the corresponding restrained qualifications for b. Hence the fourth construct qualification (4) comes into existence.

Suppose that there exists a vertex x in G such that $d_G(x, C_0) \geq \frac{1}{2}m - 1$, then we have $\gamma(x, C_0) \geq \gamma(G)$. Hence, there must exist some odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

The fifth qualification (5) thus comes into existence.

Clearly, $G \subseteq T$, then we have $G \in \mathcal{M}_{n-3}^{(2)}$.

(IV) Suppose that there exists an odd cycle C and a path $P(u, v)$ in G such that $|V(P(u, v)) \cap V(C)| = 3$. Without loss of generality,we assume that C is the smallest odd cycle which meets the above qualifications, where $V(P(u, v)) \cap V(C) = \{v_i, v_k, v_j (i < k < j)\}$. Clearly, $j = k + 1, i = k - 1$. Hence,we have

$$\begin{aligned} n - 3 &= \gamma(u, v) \leq \gamma(u, v, C) = i + (m - j) + |C| - 3 \\ &= m + |C| - 5 = |V(P(u, v) \cup C)| - 3 \leq n - 3. \end{aligned}$$

Therefore, we have

$$\gamma(u, v) = \gamma(u, v, C), |V(P(u, v)) \cup V(C)| = |V(G)| = n.$$

We might as well put the cycle $C = C_0 = y_0 y_1 \cdots y_{t_0} x_{t_0} \cdots x_1 x_0 w y_0$, where $y_0 = v_{k-1}, w = v_k$ and $x_0 = v_{k+1}$. Hence, we have

$$|C_0| = 2t_0 + 3, n = m + 2t_0 + 1.$$

We put the graph $K_{m,t_0} = P(u, v) \cup C_0$. It is easy to verify that G meets the first three construct qualifications (1), (2) and (3) of T_{m,t_0}. We shall prove that G meets the other qualifications:

Suppose that there exists an edge $x_a y_b$ in $G - E(K_{m,t_0})$ which divides C_0, where $0 \leq a, b \leq t_0$. Note that C_0 is the smallest odd cycle which has just three intersected vertexes with the (u, v)-shortest path $P(u, v)$, then we have $a + b \equiv 1 (\mathrm{mod}\ 2)$. In addition, we have

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(x_a, v_j) &+ |C_0| - d_G(x_a, v_j) - d_G(v_i, v_j) - d_G(v_i, y_b) \\ &\geq \gamma(u, v) \end{aligned}$$

and

$$\begin{aligned} d_G(v_0, v_m) + 2d_G(v_i, y_b) &+ |C_0| - d_G(x_a, v_j) - d_G(v_i, v_j) - d_G(v_i, y_b) \\ &\geq \gamma(u, v). \end{aligned}$$

Hence, we have

$$m + a + 2t_0 - b + 1 \geq m + 2t_0 - 2$$

and

$$m + b + 2t_0 - a + 1 \geq m + 2t_0 - 2.$$

We thus get $|a-b| \leq 3$.

Suppose that there exists an edge $v_k x_a$ in $G - E(K_{m,t_0})$ which divides up C_0, where $1 \leq a \leq t_0$.

If a is an odd number, then we have

$$d_G(v_0, v_m) - d_G(v_k, v_j) + 1 + d_G(v_j, x_a) - 1 \geq \gamma(u, v),$$

i.e.

$$m - 1 + 1 + a - 1 \geq m + 2t_0 - 2.$$

Thus, we have $a \geq 2t_0 - 1$. Therefore, we have $t_0 = 1, a = 1$.

If a is an even number, then we have

$$\begin{aligned} d_G(v_0, v_m) - d_G(v_i, v_k) &+ |C_0| - d_G(v_j, x_a) - d_G(v_i, v_j) + d_G(v_k, x_a) - 1 \\ &\geq \gamma(u, v), \end{aligned}$$

i.e.

$$m - 1 + (2t_0 + 3) - a - 2 + 1 - 1 \geq m + 2t_0 - 2,$$

We thus get $a \leq 2$. Hence, we have $a = 2$.

Suppose that there is an edge $v_k y_b$ in $G - E(K_{m,t_0})$ which divides C_0, where $1 \leq b \leq t_0$. Using an analogous argument, we have $b = 2$, or $b = 1$ (iff $t_0 = 1$). Hence, the fourth qualification (4) comes into existence.

Suppose that there is a vertex x in G such that $d_G(x, C) \geq \frac{1}{2}m - 2$, then we have $\gamma(x, C) \geq \gamma(G)$. Therefore, there must be an odd cycle C such that

$$\gamma(x) = \gamma(x, C) < \gamma(G),$$

i.e.

$$2d_G(x, C) + |C| - 1 < n - 3 = m + 2r.$$

Hence, the fifth qualification (5) comes into existence.

Clearly,$G \subseteq T$, then we have $G \in \mathcal{M}_{n-3}^{(3)}$. $\square$

Using the connection between the exponent of a matrix and the exponent of a graph stated above, we have get the following result by combining Theorems 4.1 with 4.2.

Theorem 4.3 *Let A be a symmetric primitive matrix with order n, then $\gamma(A) = n - 3$ iff $G(A) \in \mathcal{N}_{n-3} \cup \mathcal{M}_{n-3}$.*

References

[1] J.A.Bondy and U.S.R.Murty, *Graph Theory with Applications*, Macmillan Press, London(1976).

[2] R.A.Brualdi and H.J.Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York(1991).

[3] B.L.Liu, *Combinatorial Matrix Theory(second edition,in chinese)*, Sience Press, Beijing(2005).

[4] J.Y.Shao, The exponent set of symmetric primitive matrices, *Scientia Sinica*, A, Vol.9 (1986), pp.931-939.

[5] J.Z.Wang, The character of symmetric primitive matrices with certain exponents, *J.Taiyuan Machinery College.*, (1991)No.2.

[6] G.R.Li, The characterization of symmetric primitive matrices with exponent $2n-6$, *J.Nanjing Univ.*(Graph Theory), Vol.27(1991),pp.87-92.

[7] J.L.Cai and K.M.Zhang, The characterization of symmetric primitive matrices with exponent $2n-2r(\geq n)$, *Linear Multilin.Alg.*, 39(1995), pp.391-396.

[8] J.L.Cai and B.Y.Wang, The characterization of symmetric primitive matrices with exponent $n-1$, *Linear Alg.Appl.*, 364(2003), pp.135-145.

[9] B.L.Liu, B.D.McKay, N.C.Wormald and K.M.Zhang, The exponent set of symmetric primitive (0,1) matrices with zero trace, *Linear Alg.Appl.*, 133(1990), pp.121-131.

Math.Combin.Book.Ser. Vol.3 (2008), 79-84

The Crossing Number of the Circulant Graph $C(3k-1;\{1,k\})$

Jing Wang and Yuanqiu Huang

(Department of Mathematics, Normal University of Hunan, Changsha 410081, P.R.China)

E-mail: wangjing1001@hotmail.com, hyqq@public.cs.hn.cn

Abstract: A Smarandache drawing of a graph G is a drawing of G on the plane with minimal intersections for its each component and a circulant graph $C(n;S)$ is the graph with vertex set $V(C(n;S)) = \{v_i|0 \leqslant i \leqslant n-1\}$ and edge set $E(C(n;S)) = \{v_iv_j|0 \leq i \neq j \leq n-1,\ (i-j)\,\mathrm{mod}\, n \in S\}$, $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$. In this paper, we investigate the crossing number of the circulant graph $C(3k-1;\{1,k\})$ and get the result that $k \leq cr(C(3k-1;\{1,k\})) \leq k+1$ for $k \geqslant 3$.

Key Words: Graph, Smarandache drawing, crossing number, circulant graph.

AMS(2000): 05C, 05C62.

§1. Introduction

A *graph* $G = (V, E)$ is a set V of vertices and a subset E of unordered pairs of vertices, called edges. A Smarandache drawing of a graph G is a drawing of G on the plane with minimal intersections for its each component. Certainly, we only need to consider Smarandache drawing of connected graphs. The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges in a drawing of G in the plane. It is well known that the crossing number of a graph is attained only in *good drawings* of the graph, which are those drawings where no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr(D)$.

The *circulant graph* $C(n;S)$ is the graph with vertex set $V(C(n;S)) = \{v_i|0 \leqslant i \leqslant n-1\}$ and edge set $E(C(n;S)) = \{v_iv_j|0 \leq i \neq j \leq n-1,\ (i-j)\,\mathrm{mod}\, n \in S\}$, $S \subseteq \{1, 2, \cdots, \lfloor \frac{n}{2} \rfloor\}$.

Calculating the crossing number of a given graph is NP-complete [1]. Only the crossing number of very few families of graphs are known exactly, some of which are the crossing number of circulant graph.

Yang and Lin, etc. researched on the crossing number of circulant graphs. In [2] they showed that

$$cr(C(n;\{1,3\})) = \lfloor \frac{n}{3} \rfloor + n \,\mathrm{mod}\, 3 \quad (n \geq 8).$$

In [3], they gave an upper bound of $C(mk;\{1,k\})$ for $m \geq 3$, $k \geq 3$, proved that

$$cr(C(3k;\{1,k\})) = k \qquad (k \geq 3),$$

[1]Supported by NNSFC (10771062) and the New Century Excellent Talents in University (07-0276)

[2]Received August 16, 2008. Accepted September 20, 2008.

and in [4], they obtained that the crossing number of $C(n;\{1,\lfloor\frac{n}{2}\rfloor-1\})$ is $n/2$ for even $n\geq 8$, for odd $n\geq 13$, they showed that

$$cr(C(n;\{1,\lfloor\frac{n}{2}\rfloor-1\}))\leqslant\begin{cases}4h+2, & n=8h+1, \quad h\geqslant 2,\\ 4h+2, & n=8h+3, \quad h\geqslant 2,\\ 4h+3, & n=8h+5, \quad h\geqslant 1,\\ 4h+5, & n=8h+7, \quad h\geqslant 1.\end{cases}$$

In 2005, Ma, et al. determined that the crossing number of $C(2m+2;\{1,m\})$ is $m+1$ for $m\geq 3$, see [5].

P.T.Ho [6] investigated the crossing number of the circulant graph $C(3k+1;\{1,k\})$ and proved that $cr(C(3k+1;\{1,k\}))=k+1$ for $k\geqslant 3$.

In this paper, we study the crossing number of the circulant graph $C(3k-1;\{1,k\})$ and get the main result that

$$k\leq cr(C(3k-1;\{1,k\}))\leq k+1 \quad \text{for } k\geqslant 3.$$

§2. Some lemmas and the main result

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings made by an edge in A and another edge in B is denoted by $cr_D(A,B)$. The number of crossings made by two edges in A is denoted by $cr_D(A)$, then $cr(D)=cr_D(E)$. By counting the number of crossings in D, we have Lemma 2.1.

Lemma 2.1 *Let A,B,C be mutually disjoint subsets of E. Then*

$$cr_D(A\cup B)=cr_D(A)+cr_D(B)+cr_D(A,B);$$
$$cr_D(A\cup B,C)=cr_D(A,C)+cr_D(B,C).$$

Let $E_i=\{v_iv_{i+1},v_iv_{k+i},v_{k+i}v_{2k+i},v_{i+1}v_{2k+i},v_{k+i-1}v_{k+i},v_{2k+i-1}v_{2k+i}\}$ for $0\leqslant i\leqslant k-2$, and let $E_{k-1}=\{v_{k-1}v_{2k-1},v_{2k-1}v_0,v_{2k-2}v_{2k-1},v_{3k-2}v_0\}$, see Fig.1. Then it is not difficult to observe that

$$E(C(3k-1;\{1,k\}))=\bigcup_{i=0}^{k-1}E_i$$

$$E_i\cap E_j=\emptyset, \quad 0\leq i\neq j\leq k-1$$

We define $f_D(E_i)$ $(0\leq i\leq k-1)$ to be a function counting the number of crossings related to E_i in a drawing D as follows:

$$f_D(E_i)=cr_D(E_i)+\sum_{0\leq j\leq k-1,\, j\neq i}cr_D(E_i,E_j)/2.$$

With Lemma 2.1 and the above notations, we can get

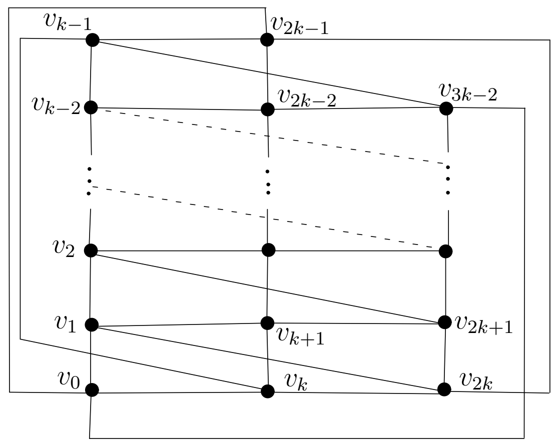


Figure 1: A good drawing of $C(3k-1;\{1,k\})$

Lemma 2.2 $cr(D) = \sum_{i=0}^{k-1} f_D(E_i)$.

In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D. The following lemma is a trivial observation.

Lemma 2.3 *If there exists a crossed edge e in a drawing D and deleting it results in a new drawing D^*, then $cr(D) \geqslant cr(D^*) + 1$.*

Lemma 2.4 $cr(C(3k-1;\{1,k\})) \geqslant k$ for $k \geqslant 3$.

Proof We will prove it by induction on k. For $k = 3$, from [2], we have $cr(C(8;\{1,3\})) = 4 \geqslant 3$. Now suppose that for $k \geqslant 4$, $cr(C(3(k-1)-1;\{1,k-1\})) \geqslant k-1$, let D be a good drawing of $C(3k-1;\{1,k\})$.

Since $C(3k-1;\{1,k\})$ is non-planar, one of the edges in D must be crossed, that is to say, v_iv_{i+1} or v_iv_{k+i} is crossed for some i where $0 \leqslant i \leqslant 3k-2$. If v_iv_{i+1} is crossed for some i, we may assume that $i = 3k-2$. If v_iv_{k+i} is crossed for some i, we may assume that $i = k-1$. By these assumptions, we have

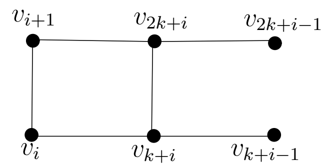
$$f_D(E_{k-1}) \geqslant 0.5$$

We assert that

$$f_D(E_i) \geq 1 \quad \text{for} \quad 0 \leqslant i \leqslant k-2 \quad \text{or} \quad cr(D) \geq k \tag{1}$$

Therefore, if $cr(D) < k$, we have $f_D(E_i) \geqslant 1$ for all $i = 0, 1, \cdots, k-2$ by (1), combining this with $f_D(E_{k-1}) \geqslant 0.5$, by Lemma 2.2, we have $k > cr(D) \geqslant k - 1 + 0.5$, which is impossible since $cr(D)$ must be an integer.

So, it suffices to verify that (1) is true. Suppose by contradiction that there exists i $(0 \leqslant i \leqslant k-2)$ such that $f_D(E_i) < 1$. From the definition of f_D, we get that $cr_D(E_i) = 0$. Furthermore, there are only two possible drawings of E_i, which are shown in Figure 2.



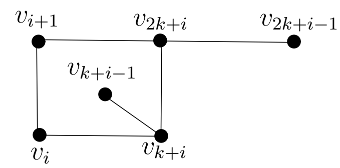


Figure 2: Two possible drawings of E_i

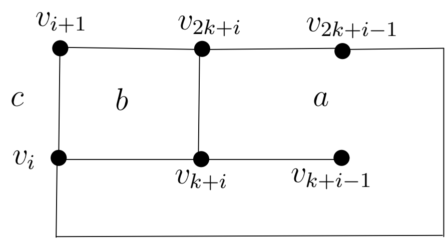


Figure 3: $E_i \cup v_i v_{2k+i-1}$

We can claim that E_i must be drawn as in the left hand side of Figure 2 in D. Suppose that E_i is drawn as in the right hand side of Figure 2. Since vertex v_{k+i-1} and vertex v_{2k+i-1} lie in different regions, so both the edge $v_{k+i-1}v_{2k+i-1}$ and the path $v_{k+i-1}v_{k+i-2}v_{2k+i-2}v_{2k+i-1}$ must cross the edges in E_i, and we have $f_D(E_i) \geqslant 1$, a contradiction to our assumption that $f_D(E_i) < 1$.

Case 1. Suppose that $f_D(E_i) > 0$. Since $f_D(E_i) < 1$, from the definition of f_D, exactly one of the edges in E_i is crossed.

First we consider that $v_i v_{2k+i-1}$ is clean. Then $E_i \cup v_i v_{2k+i-1}$ must be drawn as in Figure 3. Denote the regions by a, b and c as in Fig.3. We can assert that vertex v_{k+i-2} must lie in the same region in which vertex v_{k+i-1} lies. Or else, both the edge $v_{k+i-2}v_{k+i-1}$ and the path $v_{k+i-2}v_{i-2}v_{i-1}v_{k+i-1}$ must cross the edges on the boundary of a except $v_i v_{2k+i-1}$, so we have $f_D(E_i) \geqslant 1$, which is a contradiction. Furthermore, we can also get that v_{2k+i-2} must lie in the region a : if v_{2k+i-2} lies in the region b, then both the edge $v_{2k+i-1}v_{2k+i-2}$ and the edge $v_{k+i-2}v_{2k+i-2}$ must cross the edges on the boundary of b, which is a contradiction; if v_{2k+i-2} lies in the region c, then both the edge $v_{k+i-2}v_{2k+i-2}$ and the path $v_{k+i}v_{k+i+1}\cdots v_{2k+i-3}v_{2k+i-2}$ must cross the edges in E_i, which is also a contradiction. Since both vertex v_{k+i-2} and vertex v_{2k+i-2} lie in the region a, the pathes $v_{i+1}v_{i+2}\cdots v_{k+i-3}v_{k+i-2}$ and $v_{i+1}v_{k+i+1}v_{k+i+2}\cdots v_{2k+i-3}v_{2k+i-2}$ must cross the boundary of a, respectively, and we can have $f_D(E_i) \geqslant 1$, which is impossible.

Now consider that $v_i v_{2k+i-1}$ is crossed.

Case 1.1. Suppose that the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$ are clean. We will produce from D a drawing D^*, which is constructed by drawing a new edge connecting vertex v_{i+1} to vertex v_{2k+i-1} close enough to the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$, and by deleting the edges $v_i v_{2k+i-1}$, $v_i v_{k+i}$, $v_{k+i}v_{2k+i}$ and $v_{i+1}v_{2k+i}$, see Figure 4(1). Since the edges $v_{i+1}v_{2k+i}$ and $v_{2k+i-1}v_{2k+i}$ are clean, one can observe that the new edge $v_{i+1}v_{2k+i-1}$ doesn't produce any additional crossing. And because the crossed edge $v_i v_{2k+i-1}$ in D is removed from D,

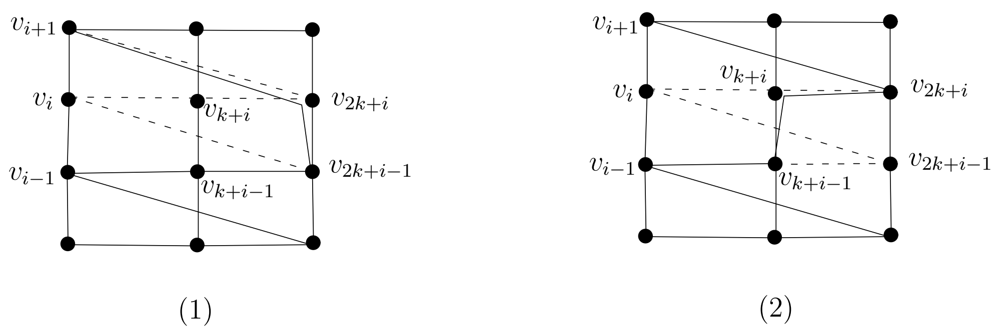


Figure 4: New drawing D^* produced from drawing D

we can get that $cr(D) \geqslant cr(D^*)+1$ by Lemma 2.3. D^* is a drawing of the subdivision of $C(3(k-1)-1;\{1,k-1\})$, so we have $cr(D) \geqslant cr(C(3(k-1)-1;\{1,k-1\}))+1 \geqslant k$.

Case 1.2. Suppose that one of the edges $v_{i+1}v_{2k+i}$ or $v_{2k+i-1}v_{2k+i}$ is crossed. Analogously, by drawing a new edge connecting vertex v_{k+i-1} to vertex v_{2k+i} quite close to the edges $v_{k+i-1}v_{k+i}$ and $v_{k+i}v_{2k+i}$, and by deleting the edges v_iv_{k+i}, $v_{k+i}v_{2k+i}$, v_iv_{2k+i-1} and $v_{k+i-1}v_{2k+i-1}$, a new drawing D^* can be produced from D, see Figure 4(2). One can easily see that the new edge $v_{k+i-1}v_{2k+i}$ doesn't produce any additional crossing since the edges $v_{k+i-1}v_{k+i}$ and $v_{k+i}v_{2k+i}$ are all clean. Since the crossed edge v_iv_{2k+i-1} in D is removed from D, by Lemma 2.3, we can obtain that $cr(D) \geqslant cr(D^*)+1$. D^* is a drawing of the subdivision of $C(3(k-1)-1;\{1,k-1\})$ as well. These facts imply that $cr(D) \geqslant cr(C(3(k-1)-1;\{1,k-1\}))+1 \geqslant k$.

Case 2. Suppose that $f_D(E_i)=0$. Since the edges in E_i are all clean, v_iv_{2k+i-1} doesn't cross any edge in E_i, then $E_i \cup v_iv_{2k+i-1}$ is drawn as in Figure 3. If v_iv_{2k+i-1} is clean, then the boundary of a is clean, we follow the analogous arguments presented in Case 1. If v_iv_{2k+i-1} is crossed, we can follow the same arguments presented in Case 1.1.

From all the above cases, we have shown that (1) is true. □

Theorem 2.5 $k \leqslant cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$.

Proof A good drawing of $C(3k-1;\{1,k\})$ in Fig.1 shows that $cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$. This together with Lemma 2.4 immediately indicate that $k \leqslant cr(C(3k-1;\{1,k\})) \leqslant k+1$ for $k \geqslant 3$. □

We end this paper with the following conjecture.

Conjecture $cr(C(3k-1;\{1,k\})) = k+1$ for $k \geqslant 3$.

References

[1] Garey, M. R., Johnson, D. S, Crossing number is NP-complete, *SIAM J.Algebraic Discrete Methods*, 4(1983), 312-316.

[2] Yang, Y., Lin, X., Lu, J., Hao, X., The crossing number of $C(n;\{1,3\})$, *Discrete Math.*, 289(2004), 107-118.

[3] Lin, X., Yang, Y., Lu, J., Hao, X., The crossing number of $C(mk; \{1, k\})$, *Graphs and Combinatorics*, 21(2005), 89-96.

[4] Lin, X., Yang, Y., Lu, J., Hao, X., The crossing number of $C(n; \{1, \lfloor \frac{n}{2} \rfloor - 1\})$, *Util. Math.*, 71(2006), 245-255.

[5] Ma, D., Ren, H., Lu, J., The crossing number of the circular graph $C(2m+2, m)$, *Discrete Math.*, 304(2005), 88-93.

[6] Pak Tung Ho, The crossing number of $C(3k+1; \{1, k\})$, *Discrete Math.*, 307(2007), 2771-2774.

Math.Combin.Book.Ser. Vol.3 (2008), 85-93

On the Edge Geodetic and k-Edge Geodetic Number of a Graph

A.P. Santhakumaran and S.V. Ullas Chandran

(Department of Mathematics of St.Xavier's College (Autonomous), Palayamkottai - 627 002, Tamil Nadu, India.)

E-mail: apskumar1953@yahoo.co.in, ullaschandra01@yahoo.co.in

Abstract: For vertices u and v in a connected graph $G = (V, E)$, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For an integer $k \geq 1$, a geodesic of length k in G is called a k-geodesic. A set $S \subseteq V$ is a k-edge geodetic set of G if each edge $e \in E - E(< S >)$ lies on a k-geodesic of some pair of vertices in S and a set $T \subseteq V$ is an *edge geodetic set* of G if each edge of G lies on a geodesic of some pair of vertices in T, and Smarandache edge-geodetic set of G if each edge of G lies on at least two geodesics of T. The minimum cardinality of a k-edge geodetic set of G is the *k-edge geodetic number* $eg_k(G)$ and the minimum cardinality of an edge geodetic set is the *edge geodetic number* $eg(G)$. In this paper we investigate how the edge geodetic number and the k-edge geodetic number of a graph G are affected by adding a pendant edge to G. It is proved that if G' is a graph obtained from G by adding a pendant edge, then $eg(G) \leq eg(G') \leq eg(G)+1$ and $eg_2(G) \leq eg_2(G') \leq eg_2(G)+1$. For any integer $k \geq 2$, it is also proved that $eg_k(G') \leq eg_k(G) + 2$. It is shown that for any integer $k \geq 4$ and for every pair a, b of integers with $4 \leq a \leq b + 2$, there is a connected graph G such that $eg_k(G) = b$ and $eg_k(G') = a$, where G' is a graph obtained from G by adding a pendant edge.

Key Words: Smarandache edge-geodetic set, geodetic number, edge geodetic number, k-geodetic number, k-edge geodetic number, k-extreme edge.

AMS(2000): 05C12.

§1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to Harary [3]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest u-v path in G. It is known that the distance is a metric on the vertex set of G. A u-v path of length $d(u, v)$ is called a u-v *geodesic*. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For any path P in a graph and two vertices x, y on P, we use $P[x, y]$ to denote the portion of P between x and y, inclusive of x and y. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. The *closed interval* $I[u, v]$ consists of all vertices lying on

[1]Supported by DST Project No. SR/S4/MS: 319/06

[2]Received May 6, 2008. Accepted September 2, 2008.

some u-v geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{u,v \in S} I[u,v]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. The geodetic number of a graph was introduced in [1], [4] and further studied in [2], [5]. It was shown in [4] that determining the geodetic number of a graph is an NP-hard problem. A set S of vertices is an *edge geodetic set* of a graph G if each edge of G lies on a geodesic of vertices in S, and Smarandache edge-geodetic set of G if each edge of G lies on at least two geodesics of S. The minimum cardinality of an edge geodetic set is the *edge geodetic number* $eg(G)$. An edge geodetic set of cardinality $eg(G)$ is called *eg-set* of G. Edge geodetic sets and the edge geodetic number of a graph with several interesting applications are investigated in [7].

For an integer $k \geq 1$, a geodesic in G of length k is called *k-geodesic*. A vertex v is called *k-extreme vertex* if v is not the internal vertex of a k-geodesic joining any pair of distinct vertices of G. Obviously, each extreme vertex of a connected graph G is k-extreme vertex of G. In particular, each end vertex of G is a k-extreme vertex of G. A set $S \subseteq V$ is called a *k-geodetic set* of G if each vertex v in $V - S$ lies on a k-geodesic of vertices in S. The minimum cardinality of a k-geodetic set of G is its *k-geodetic number* $g_k(G)$. A k-geodetic set of cardinality $g_k(G)$ is called g_k*-set*. The k-geodetic number of a graph was refereed to as *k-geodeomination number* and studied in [6].

For any $S \subseteq V$, let $E(< S >)$ denote the edge set of the subgraph induced by S. A set $S \subseteq V$ is called a *k-edge geodetic set* of G if each edge in $E - E(< S >)$ lies on a k-geodesic of vertices in S. The minimum cardinality of a k-edge geodetic set of G is its *k-edge geodetic number* $eg_k(G)$. A k-edge geodetic set of cardinality $eg_k(G)$ is called eg_k*-set* of G. For $k \geq 2$, an edge of G is called *k-extreme edge* if it does not lie on any k-geodesic of vertices of G.

For the graph G given in Fig.1.1, it is easy to see that the set $S = \{v_1, v_2, v_5, v_6\}$ of end vertices is a g_2-set and so $g_2(G) = 4$. Since the edge v_3v_4 does not lie on any 2-geodesic of vertices of S, S is not a 2-edge geodetic set of G. It is easily seen that $S_1 = \{v_1, v_2, v_3, v_5, v_6\}$ is a minimum 2-edge geodetic set of G so that $eg_2(G) = 5$. Also, $S_2 = \{v_1, v_2, v_4, v_5, v_6\}$ is another eg_2-set of G.

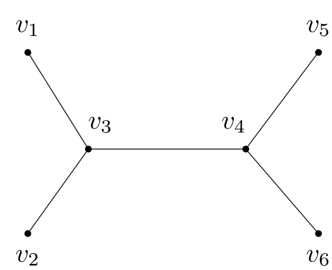


Fig.1.1

The k-edge geodetic number of a graph was introduced and studied in [8]. It is proved in [8] that each triple a, b, k of integers with $2 \leq a \leq b$ and $k \geq 2$ is realizable as the k-geodetic number and k-edge geodetic number of a graph respectively. Also it is shown in [8] that for given integers a, b, c and $k \geq 2$ with $3 \leq a \leq b \leq c$, there is a connected graph G with

$g(G) = a$, $eg(G) = b$ and $eg_k(G) = c$. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design.

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. The geodetic number and the k-geodetic number, affected by adding a pendant edge, was discussed in [5] and [6] respectively. In this paper we study how the edge geodetic number and the k-edge geodetic number of a graph are affected by adding a pendant edge to the graph.

Throughout the following G denotes a connected graph with at least two vertices. The following theorems will be used in the sequel.

Theorem 1.1([7]) *Each extreme vertex of a connected graph G belongs to every edge geodetic set of G. In particular, if the set of all extreme vertices W is an edge geodetic set of G, then W is the unique eg-set of G.*

Theorem 1.2([8]) *Every k-edge geodetic set contains both the ends of each k-extreme edge. If the set W of the ends of all the k-extreme edges together with the set of k-extreme vertices is a k-edge geodetic set, then W is the unique eg_k-set of G and so $eg_k(G) = |W|$.*

Theorem 1.3([7]) *For any tree T with k end vertices, $eg(T) = k$.*

Theorem 1.4([7]) *For a connected graph G, $eg(G) = 2$ if and only if there exist two antipodal vertices u and v such that every edge lies on a $u - v$ geodesic of G.*

Theorem 1.5([7]) *For a connected graph G, no cut vertex belongs to any eg-set of G.*

Theorem 1.6([7]) *For the complete bipartite graph $K_{m,n}(m, n \geq 2)$, $eg(K_{m,n}) = min\{m, n\}$.*

Theorem 1.7([7]) *If a connected graph G of order n has exactly one vertex v of degree $n - 1$ then $eg(G) = n - 1$.*

§2. How the edge geodetic number of a connected graph is affected by adding a pendant edge

In this section we discuss how the edge geodetic number of a connected graph G is affected by adding a pendant edge to G. Let G' be a graph obtained from a connected graph G by adding a pendant edge uv, where u is not vertex of G and v is a vertex of G.

Theorem 2.1 *If G' is a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G, then $eg(G) \leq eg(G') \leq eg(G) + 1$.*

Proof Let S be any eg-set of G and let $S' = S \cup \{u\}$. We claim that S' is an edge geodetic set of G'. Let e be an edge of G'. If $e \in E(G)$, then e lies on a geodesic of vertices in S. If $e = uv$, then, since every edge geodetic set of G is a geodetic set of G, it follows that the vertex v lies on a $x - y$ geodesic P with $x, y \in S$. Then, it is clear that the portion $P[x, v]$ of the $x - v$ path on P together with the edge uv is a $x - u$ geodesic of G', which contains the edge e with

$x, u \in S'$. Hence S' is an edge geodetic set of G' and so $eg(G') \leq eg(G)+1$. Let S' be an eg-set of G'. By Theorems 1.1 and 1.5, $u \in S'$ and $v \notin S'$. Also, it is clear that $S = (S' - \{u\}) \cup \{v\}$ is an edge geodetic set of G so that $eg(G) \leq |S'| - 1 + 1 = |S'| = eg(G')$. Hence the result. □

Remark 2.2 The bounds for $eg(G')$ in Theorem 2.1 are sharp. If the graph G is the path $P_n (n \geq 3)$ on n vertices, then, by Theorem 1.3, $eg(P_n) = 2$. Let G' be the path obtained from P_n by adding a pendant edge at one of its end vertices. Then, by Theorem 1.3, $eg(G') = 2 = eg(G)$. If G' is the tree obtained from P_n by adding a pendant edge at a cut vertex of P_n, then by Theorem 1.3, $eg(G') = 3 = eg(G) + 1$.

Theorem 2.3 *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $eg(G) = eg(G')$ if and only if v is a vertex of some eg-set of G.*

Proof First, assume that there is an eg-set S of G such that $v \in S$. Let $S' = (S - \{v\}) \cup \{u\}$. We show that S' is an eg-set of G'. If $e = uv$, then it is clear that e lies on every $w - u$ geodesic of G, where $w \in S' (w \neq u)$. Let e be any edge of G. Since S is an eg-set of G, e lies on a $x - y$ geodesic in G with $x, y \in S$. If both $x, y \in S - \{v\}$, then e also lies on a $x - y$ geodesic in G' with $x, y \in S'$. If e lies on a $x - v$ geodesic in G with $x \in S - \{v\}$, then e also lies on $x - u$ geodesic in G'. Thus S' is an edge geodetic set of G' so that $eg(G') \leq |S'| = |S| = eg(G)$. Now, the result follows from Theorem 2.1.

Conversely, suppose that $eg(G) = eg(G')$. Suppose that v does not belong to any eg-set of G. Let S' be an eg-set of G'. Since u is an end vertex of G' and v is a cut vertex of G', by Theorems 1.1 and 1.5, $u \in S'$ and $v \notin S'$. Let $S = (S' - \{u\}) \cup \{v\}$. Then $S \subseteq V(G)$ and $|S| = |S'| = eg(G') = eg(G)$. Let e be any edge of G. Then e is also an edge of G' and so e lies on a geodesic P in G' joining a pair of vertices $x, y \in S'$. If $x \neq u$ and $y \neq u$, then $x \in S$ and $y \in S$ so that e lies on a geodesic joining a pair of vertices in S. Otherwise, let $x \neq u$ and $y = u$. Then it follows that e lies on a geodesic in G joining x and v in S. Thus, S is an edge geodetic set of G and since $|S| = eg(G)$, it follows that S is an eg-set of G. Since $v \in S$, this is contradiction to our assumption. This completes the proof. □

Remark 2.4 If a vertex v is added to a connected graph G such that more than one edge is incident with v, then the edge geodetic number of the resulting graph can stay the same, increase significantly or decrease significantly. For example, for the complete bipartite graph $K_{m,n}$ we have, by Theorem 1.6, $eg(K_{m,n}) = m$ for all $2 \leq m \leq n$. However, if we add a new vertex to $K_{m,n}$ and join this vertex to all the vertices of the minimum partite set containing m vertices, the resulting graph is $K_{m,n+1}$ and again by Theorem 1.6, the edge geodetic number is m. Hence a new vertex may be added to a graph along with a large number of edges such that it does not affect the edge geodetic number. On the other hand, it is clear that $eg(C_n) = 2$ for all even $n \geq 4$. If we add a vertex v to this C_n and join v to all the vertices of C_n, the resulting graph is the wheel $K_1 + C_n$. Now, it follows from Theorem 1.7 that $eg(K_1 + C_n) = n$ and so the edge geodetic number of the resulting graph increases significantly. Also, it is clear that $eg(K_{1,n}) = n$ for all $n \geq 2$. If we add a vertex v and join it to all the end vertices of $K_{1,n}$ then we obtain the graph $K_{2,n}$. By Theorem 1.6, $eg(K_{2,n}) = 2$, and so the edge geodetic number of the resulting graph decreases significantly for large n.

§3. How the k-edge geodetic number of a connected graph is affected by adding a pendant edge

We now consider how the k-edge geodetic number of a connected graph G is affected by the addition of a pendant edge.

Proposition 3.1 *Let G' be a graph obtained from a connected graph G by adding a pendant edge uv at a vertex v of G. Then $eg_k(G') \leq eg_k(G) + 2$.*

Proof Let S be an eg_k-set of G. Then $S \cup \{u, v\}$ is a k-edge geodetic set of G' and so $eg_k(G') \leq |S \cup \{u, v\}| \leq eg_k(G) + 2$. □

Proposition 3.2 *There is no connected graph G with $diam(G) \geq k$ such that $eg_k(G') = 2$, where G' is a graph obtained from G by adding a pendant edge at a vertex of G.*

Proof Suppose that there exists a connected graph G with $diam(G) \geq k$ such that $eg_k(G') = 2$. Let G' be a graph obtained from G by adding a pendant edge uv at a vertex v of G. By Theorem 1.1, u belongs to every edge geodetic set of G'. Let $S' = \{u, y\}$ be an eg_k-set of G'. Then $y \neq v$ and it is clear that $S = \{v, y\}$ is a eg_{k-1}- set of G. Hence S is an eg-set of G and $d(v, y) = k - 1$. Now, by Theorem 1.4, v and y are antipodal vertices and so $diam(G) = k - 1$, which is a contradiction. Hence the result follows. □

Observation 3.3 In a connected graph G, each edge in G has at least one end in every 2-edge geodetic set of G.

Theorem 3.4 *If G' is a graph obtained from a connected graph G by adding a pendant edge at a vertex of G, then $eg_2(G) \leq eg_2(G') \leq eg_2(G) + 1$.*

Proof Let G' be the graph obtained from G by adding a pendant edge uv at a vertex v of G. Let S be an eg_2-set of G. Let $S' = S \cup \{u\}$. We claim that S' is a 2-edge geodetic set of G'. Let e be an edge of G' be such that $e \notin E(< S' >)$. If $e \in E(G)$, then e lies on a 2-geodesic of vertices in S. If $e \notin E(G)$, then $e = uv$ and $v \notin S$. Let vw be an edge of G. Then, by Observation 3.3, we have $w \in S$. Now, it is clear that the edge uv lies on the 2-geodesic $P : w, v, u$ of G' with $w, u \in S'$. Hence S' is a 2-edge geodetic set of G' and so $eg_2(G') \leq |S'| = eg_2(G) + 1$.

Now, let T' be any eg_2-set of G'. Then by Theorem 1.2, $u \in T'$. Let $T = (T' - \{u\}) \cup \{v\}$. Then $|T| \leq |T'|$. We show that T is a 2-edge geodetic set of G. Let $e = xy$ be any edge of G such that $e \notin E(< T >)$. Then it is clear that $e \notin E(< T' >)$. Now, since $e \in E(G')$ and T' is a eg_2-set of G', we see that $e = xy$ lies on a 2-geodesic P of vertices in T'. By Observation 3.3, we may assume that $x \in T'$. Assume that the geodesic P is $P : x, y, z$ with $x, z \in T'$. Since $xy \in E(G)$, we have $x \neq u$ and so $x \in T$. Now, if $z = u$, then $y = v$ and so $xy \in E(< T >)$, which is a contradiction. Hence $z \neq u$ and so $z \in T$. Thus T is a 2-edge geodetic set of G so that $eg_2(G) \leq |T| \leq |T'| = eg_2(G')$. □

Proposition 3.5 *Let G' be a graph obtained from a connected graph G by adding a pendant*

edge uv at a vertex v of G. If v belongs to some eg_2-set of G', then $eg_2(G') = eg_2(G) + 1$.

Proof Let T' be an eg_2-set of G' such that $v \in T'$. By Theorem 1.2, $u \in T'$. Now, let $T = T' - \{u\}$. Then $|T| = |T'| - 1 = eg_2(G') - 1$ and as in the proof of Theorem 3.4, T is a 2-edge geodetic set of G so that $eg_2(G) \leq |T| = eg_2(G') - 1$ Now, the result follows from Theorem 3.4. □

Remark 3.6 The converse of Theorem **??** is not true. For the graph $G = K_{1,n} (n \geq 2)$, we have that $eg_2(G) = n$. However, if we add a pendant edge to the cut vertex of $K_{1,n}$, then the resulting graph G' is $K_{1,n+1}$ and so $eg_2(G') = n + 1 = eg_2(G) + 1$. However, the cut vertex of $K_{1,n+1}$ does not belong to any eg_2-set of $K_{1,n+1}$.

Problem 3.7 *Characterize graphs G for which $eg_2(G') = eg_2(G)$, where G' is a graph obtained from G by adding a pendant edge.*

In view of Proposition 3.1, we have the following realization theorem.

Theorem 3.8 *Let $k \geq 4$ be an integer. For each pair a, b of integers with $4 \leq a \leq b + 2$, there is a connected graph G with $eg_k(G) = b$ and $eg_k(G') = a$, where G' is a graph obtained from G by adding a pendant edge.*

Proof We prove the theorem by considering five cases.

Case 1. Let $a = b$. Let G be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding $b - 3$ new vertices $u_1, u_2, \ldots, u_{b-3}$ and joining them to v_2. The graph G is shown in Fig.3.1. It is clear that the edges $u_i v_2 (1 \leq i \leq b - 3)$ are the only k-extreme edges of G. Hence $S = \{v_0, v_k, u_1, u_2, \ldots, u_{b-3}, v_2\}$ is the set of all k-extreme vertices and the ends of all k-extreme edges of G. Since S is a k-edge geodetic set of G, it follows from Theorem 1.2 that $eg_k(G) = |S| = b$.

Now, let G' be the graph obtained from G by adding a pendant edge $v_k x$. It is clear that G' has no k-extreme edges. Let $S' = \{v_0, u_1, u_2, \ldots, u_{b-3}, x\}$ be the set of all k-extreme vertices of G'. Since the edges $v_0 v_1$ and $v_1 v_2$ do not lie on any k-geodesic joining a pair of vertices in S', we have S' is not a k-edge geodetic set of G'. Since $S' \cup \{v_k\}$ is a k-edge geodetic set of G', it follows from Theorem 1.2 that $eg_k(G') = |S'| + 1 = b = a$.

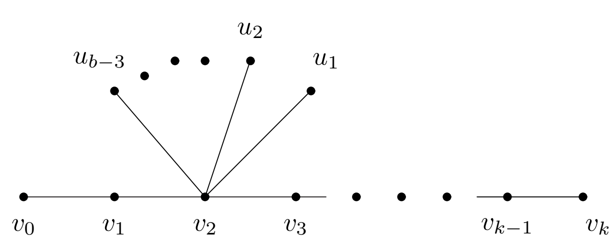


Fig.3.1

Case 2. $a = b + 1$. Let G be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding $b - 2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and joining each u_i to v_1. Let $S = \{u_1, u_2, \ldots, u_{b-2}, v_0, v_k\}$.

Then S is the set of all k-extreme vertices of G. It is clear that G has no k-extreme edges and S is a k-edge geodetic set of G and so by Theorem 1.2, $eg_k(G) = |S| = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to v_1. Then, just as above, the set $S' = \{u_1, u_2, \ldots, u_{b-2}, x, v_0, v_k\}$ of all k-extreme vertices of G' is the eg_k-set of G'. Hence $eg_k(G') = |S'| = b + 1 = a$.

Case 3. $a = b+2$. Let G be the graph constructed in Case 2. Then, as in Case 2, $eg_k(G) = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to v_2. Then the edge xv_2 is the only k-extreme edge in G'. Since the set $S' = \{u_1, u_2, \ldots, u_{b-2}, v_0, v_k, x, v_2\}$ of all k-extreme vertices together with the ends of the k-extreme edge xv_2 is a k-edge geodetic set, it follows from Theorem 1.2 that $eg_k(G') = |S'| = b + 2 = a$.

Case 4. $a = b - 1$. Let G_1 be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding a new vertex w and joining it to the vertices v_1, v_2 and v_3. Let $Q : x_0, x_1, \ldots, x_{k-2}$ be a path such that it is vertex disjoint with G_1. Let G_2 be the graph obtained from G_1 and Q by identifying the vertices v_2 and x_0. Let G be the graph obtained from G_2 by adding $b - 5$ new vertices $z_1, z_2, \ldots, z_{b-5}$ and joining each z_i to v_1. The graph is G shown in Fig.3.2. It is clear that the edge v_2w is the only k-extreme edge of G. Since the set $S = \{v_0, z_1, z_2, \ldots, z_{b-5}, v_k, x_{k-2}, v_2, w\}$ of all k-extreme vertices and the ends of the k-extreme edge v_2w of G is a k-edge geodetic set, it follows from Theorem 1.2 that $eg_k(G) = |S| = b$.

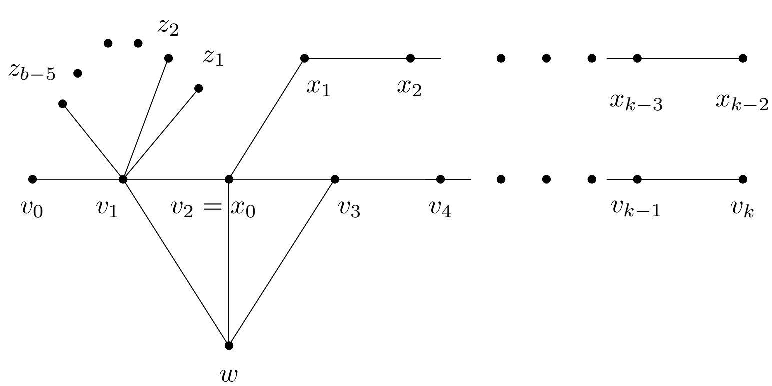


Fig.3.2

Now, let G' be the graph obtained from G by adding a pendant edge wx. It is clear that G' has no k-extreme edges. Since the set $S' = \{v_0, z_1, z_2, \ldots, z_{b-5}, x, v_k,$
$x_{k-2}\}$ of all k-extreme vertices of G' is a k-edge geodetic set of G', it follows from Theorem 1.2 that $eg_k(G') = |S'| = b - 1 = a$.

Case 5. $4 \leq a \leq b - 2$. Let G_1 be the graph obtained from the path $P : v_0, v_1, \ldots, v_k$ by adding a new vertex w and joining it to both v_2 and v_4. Let G_2 be the graph obtained from G_1 by adding $b - a - 1$ new vertices $u_1, u_2, \ldots, u_{b-a-1}$ and joining each u_i to the vertices v_2, v_3, v_4 and w. Let G_3 be the graph obtained from G_2 by adding $a - 4$ new vertices $z_1, z_2, \ldots, z_{a-4}$ and joining each z_i to v_1. Let $Q : x_0, x_1, \ldots, x_{k-3}$ be a path such that it is vertex disjoint with G_3. Let G be the graph obtained from G_3 and Q by identifying the vertices v_3 and x_0. The

graph G is shown in Fig.3.3. It is clear that the edges u_iv_3 and u_iw $(1 \leq i \leq b-a-1)$ are the only k-extreme edges of G and so by Theorem 1.2, the vertices $u_1, u_2, \ldots, u_{b-a-1}, v_3, w$ belong to every k-edge geodetic set of G.

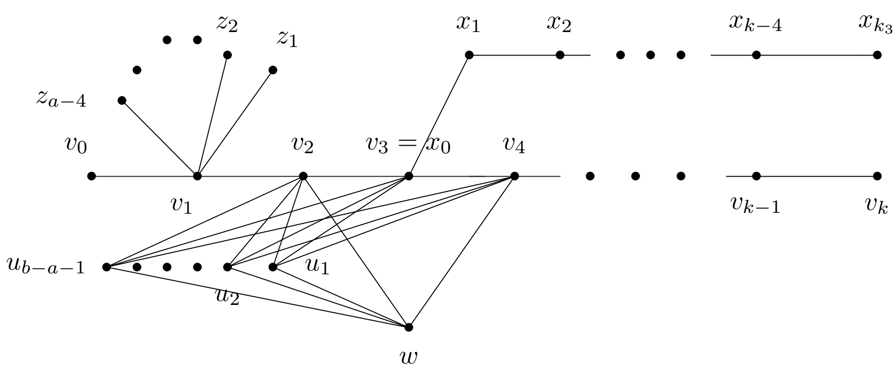


Fig.3.3

First, suppose that $k = 4$. Let $S = \{v_0, u_1, u_2, \ldots, u_{b-a-1}, v_3, w, z_1, z_2, \ldots, z_{a-4}, x_1\}$. Then S is the set of all k-extreme vertices and the ends of all k-extreme edges of G. It is clear that S is not a k-edge geodetic set of G and $S \cup \{v_4\}$ is a k-edge geodetic set of G so that by Theorem 1.2, $eg_k(G) = |S| + 1 = b - 1 + 1 = b$. Now, let G' be the graph obtained from G by adding a new vertex x and joining it to w. Then the graph G' has no k-extreme edges. Let $S' = \{v_0, z_1, z_2, \ldots, z_{a-4}, x, x_1\}$. Then S' is the set of all k-extreme vertices of G'. It is clear that S' is not a k-edge geodetic set of G' and $S' \cup \{v_4\}$ is a k-edge geodetic set of G' so that by Theorem 1.2, $eg_k(G') = |S'| + 1 = a - 1 + 1 = a$. Next, suppose that $k \geq 5$. Let $T = \{v_0, u_1, u_2, \ldots, u_{b-a-1}, v_3, w, z_1, z_2, \ldots, z_{a-4}, x_{k-3}, v_k\}$. Then T is the set of all k-extreme vertices and the ends of all k-extreme edges of G. It is clear that T is a k-edge geodetic set of G and so by Theorem 1.2, $eg_k(G) = |T| = b$.

Let G' be the graph obtained from G by adding a new vertex x and joining it to w. Then G' has no k-extreme edges and $T' = \{v_0, z_1, z_2, \ldots, z_{a-4}, x, x_{k-3}, v_k\}$ is the set of all k-extreme vertices of G'. Since T' is a k-edge geodetic set of G', it follows from Theorem 1.2 that $eg_k(G') = |T'| = a$. Thus the proof is complete. □

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990.

[2] G. Chartrand, F. Harary and P. Zhang, On the Geodetic Number of a Graph, *Networks*, **39**(1)(2002), 1-6.

[3] F.Harary, *Graph Theory,* Addison-Wesley, 1969.

[4] F. Harary, E. Loukakis, C. T. Souros, The geodetic number of a graph, *Mathl. Comput.Modeling*,**17**(11) (1993), 89-95.

[5] R. Muntean, P. Zhang, On Geodomination in Graphs, *Congr. Numer.*, 143 (2000),161-174.

[6] R. Muntean, P. Zhang, k-Geodomination in Graphs, *ARS Combinatoria*, 63 (2002), 33-47.

[7] A. P. Santhakumaran and J. John, Edge Geodetic Number of a Graph, *Journal of Discrete Mathematical Sciences & Cryptography*, **10**(3) (2007),415-432.

[8] A. P. Santhakumaran and S. V. Ullas Chandran, The k-edge geodetic number of a graph, (communicated).

Math.Combin.Book.Ser. Vol.3 (2008), 94-104

Simple Path Covers in Graphs

S. Arumugam and I. Sahul Hamid

National Centre for Advanced Research in Discrete Mathematics of

Kalasalingam University, Anand Nagar, Krishnankoil-626190, INDIA.

E-mail: *s_arumugam_akce@yahoo.com*

Abstract: A simple path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ and any two paths in ψ have at most one vertex in common. More generally, for any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G. The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. In this paper we initiate a study of this parameter.

Key Words: Smarandache path k-cover, simple path cover, simple path covering number.

AMS(2000): 05C35, 05C38.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [5]. All graphs in this paper are assumed to be connected and non-trivial.

If $P = (v_0, v_1, v_2, \ldots, v_n)$ is a path or a cycle in a graph G, then $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_n)$ and $Q = (v_n = w_0, w_1, w_2, \ldots, w_m)$ are two paths in G, then the walk obtained by concatenating P and Q at v_n is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_2, v_1, v_0)$ is denoted by P^{-1}. For a unicyclic graph G with cycle C, if w is a vertex of degree greater than 2 on C with $deg\ w = k$, let $e_1, e_2, \ldots, e_{k-2}$ be the edges of $E(G) - E(C)$ incident with w. Let $T_i, 1 \leq i \leq k-2$, be the maximal subtree of G such that T_i contains the edge e_i and w is a pendant vertex of T_i. Then $T_1, T_2, \ldots, T_{k-2}$ are called the *branches* of G at w. Also the maximal subtree T of G such that $V(T) \cap V(C) = \{w\}$ is called the *subtree* rooted at w.

The concept of path cover and path covering number of a graph was introduced by Harary [6]. Preliminary results on this parameter were obtained by Harary and Schwenk [7], Peroche [9] and Stanton et al. [10], [11].

[1]Supported by NBHM Project 48/2/2004/R&D-II/7372.

[2]Received May 16, 2008. Accepted September 16, 2008.

Definition 1.1([6]) *A path cover of a graph G is a collection ψ of paths in G such that every edge of G is in exactly one path in ψ. The minimum cardinality of a path cover of G is called the path covering number of G and is denoted by $\pi(G)$ or simply π.*

Theorem 1.2([10]) *For any tree T with k vertices of odd degree, $\pi(T) = \frac{k}{2}$.*

Theorem 1.3([7]) *The path covering number of the complete graph K_p is given by $\pi(K_p) = \lceil \frac{p}{2} \rceil$. (For any real number x, $\lceil x \rceil$ denotes the least positive integer $\geq x$.)*

Theorem 1.4([4]) *Let G be a unicyclic graph with unique cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k denote the number of vertices of odd degree. Then*

$$\pi(G) = \begin{cases} 2 & \text{if } m = 0 \\ \frac{k}{2} + 1 & \text{if } m = 1 \\ \frac{k}{2} & \text{otherwise} \end{cases}$$

Theorem 1.5([4]) *For any graph G, $\pi(G) \geq \lceil \frac{\Delta}{2} \rceil$.*

The concepts of graphoidal cover and acyclic graphoidal cover were introduced by Acharya et al. [1] and Arumugam et al. [4].

Definition 1.6([1]) *A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G satisfying the following conditions.*

(i) *Every path in ψ has at least two vertices.*
(ii) *Every vertex of G is an internal vertex of at most one path in ψ.*
(iii) *Every edge of G is in exactly one path in ψ.*

If further no member of ψ is a cycle in G, then ψ is called an acyclic graphoidal cover of G. The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$. Similarly we define the acyclic graphoidal covering number $\eta_a(G)$.

An elaborate review of results in graphoidal covers with several interesting applications and a large collection of unsolved problems is given in Arumugam et al.[2].

For any graph $G = (V, E)$, $\psi = E$ is trivially an acyclic graphoidal cover and has the interesting property that any two paths in ψ have at most one vertex in common. Motivated by this observation we introduced the concept of simple acyclic graphoidal covers in graphs [3].

Definition 1.7([3]) *A simple acyclic graphoidal cover of a graph G is an acyclic graphoidal cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple acyclic graphoidal cover of G is called the simple acyclic graphoidal covering number of G and is denoted by $\eta_{as}(G)$ or simply η_{as}.*

Definition 1.8 *Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be an interior vertex of ψ if it is an internal vertex of some path in ψ, otherwise it is said to*

be an exterior vertex of ψ.

Theorem 1.9([3]) *For any simple acyclic graphoidal cover ψ of a graph G, let t_ψ denote the number of exterior vertices of ψ. Let $t = min\ t_\psi$, where the minimum is taken over all simple acyclic graphoidal covers ψ of G. Then $\eta_{as}(G) = q - p + t$.*

Theorem 1.10([3]) *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let m denote the number of vertices of degree greater than 2 on C. Then*

$$\eta_{as}(G) = \begin{cases} 3 & \text{if } m = 0 \\ n+2 & \text{if } m = 1 \\ n+1 & \text{if } m = 2 \\ n & \text{if } m \geq 3 \end{cases}$$

Theorem 1.11([3]) *Let m and n be integers with $n \geq m \geq 4$. Then*

$$\eta_{as}(K_{m,n}) = \begin{cases} mn - m - n & \text{if } n \leq \binom{m}{2} \\ & \\ mn - m - n + r & \text{if } n = \binom{m}{2} + r, r > 0. \end{cases}$$

In this paper we introduce the concept of simple path cover and simple path covering number π_s of a graph G and initiate a study of this parameter. We observe that the concept of simple path cover is a special case of Smarandache path k-cover [8]. For any integer $k \geq 1$, a Smarandache path k-cover of a graph G is a collection ψ of paths in G such that each edge of G is in at least one path of ψ and two paths of ψ have at most k vertices in common. Thus if $k = 1$ and every edge of G is in exactly one path in ψ, then a Smarandache path k-cover of G is a simple path cover of G.

§2. Main results

Definition 2.1 *A simple path cover of a graph G is a path cover ψ of G such that any two paths in ψ have at most one vertex in common. The minimum cardinality of a simple path cover of G is called the simple path covering number of G and is denoted by $\pi_s(G)$. Any simple path cover ψ of G for which $|\psi| = \pi_s(G)$ is called a minimum simple path cover of G.*

Example 2.2 Consider the graph G given in Fig.2.1.

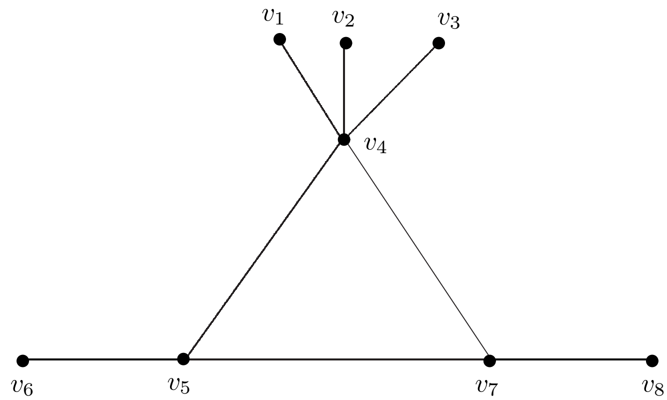


Fig. 2.1

Then $\psi = \{(v_1, v_4, v_7, v_8), (v_3, v_4, v_5, v_6), (v_2, v_4), (v_7, v_5)\}$ is a minimum simple path cover of G so that $\pi_s(G) = 4$.

Remark 2.3 Every path in a simple path cover of a graph G is an induced path.

Theorem 2.4 *For any simple path cover ψ of a graph G, let $t_\psi = \sum_{P\in\psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = max\ t_\psi$, where the maximum is taken over all simple path covers ψ of G. Then $\pi_s(G) = q - t$.*

Proof Let ψ be any simple path cover of G. Then

$$q = \sum_{P\in\psi} |E(P)|$$

$$= \sum_{P\in\psi} (t(P) + 1)$$

$$= |\psi| + \sum_{P\in\psi} t(P)$$

$$= |\psi| + t_\psi$$

Hence $|\psi| = q - t_\psi$ so that $\pi_s(G) = q - t$. □

Corollary 2.5 *For any graph G with k vertices of odd degree $\pi_s(G) = \frac{k}{2} + \sum_{v\in V(G)} \left\lfloor \frac{deg\ v}{2} \right\rfloor - t$.*

Proof Since $q = \frac{k}{2} + \sum_{v\in V(G)} \left\lfloor \frac{deg\ v}{2} \right\rfloor$ the result follows. □

Corollary 2.6 *For any graph G, $\pi_s(G) \geq \frac{k}{2}$ where k is the number of vertices of odd degree in G. Further, the following are equivalent.*

(i) $\pi_s(G) = \frac{k}{2}$.

(ii) There exists a simple path cover ψ of G such that every vertex v in G is an internal vertex of $\left\lfloor \frac{deg\ v}{2} \right\rfloor$ paths in ψ.

(ii) There exists a simple path cover ψ of G such that every vertex of odd degree is an

external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ.

Remark 2.7 For any (p,q)-graph G, $\pi_s(G) \leq q$. Further, equality holds if and only if G is complete. Hence it follows from Theorem 1.3 that $\pi_s(K_n) = \pi(K_n)$ if and only if $n = 2$.

Remark 2.8 Since any path cover of a tree T is a simple path cover of T, it follows from Theorem 1.2 that $\pi_s(T) = \pi(T) = \frac{k}{2}$, where k is the number of vertices of odd degree in T.

We now proceed to determine the value of π_s for unicyclic graphs and wheels.

Theorem 2.9 *Let G be a unicyclic graph with cycle C. Let m denote the number of vertices of degree greater than 2 on C. Let k be the number of vertices of odd degree. Then*

$$\pi_s(G) = \begin{cases} 3 & \text{if } m = 0 \\ \frac{k}{2} + 2 & \text{if } m = 1 \\ \frac{k}{2} + 1 & \text{if } m = 2 \\ \frac{k}{2} & \text{if } m \geq 3 \end{cases}$$

Proof Let $C = (v_1, v_2, \ldots, v_r, v_1)$.

Case 1. $m = 0$.

Then $G = C$ so that $\pi_s(G) = 3$.

Case 2. $m = 1$.

Let v_1 be the unique vertex of degree greater than 2 on C. Let G_1 be the tree rooted at v_1. Then G_1 has k vertices of odd degree and hence $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1.

If $deg\ v_1$ is odd, then $deg_{G_1}\ v_1$ is odd. Let P be the path in ψ_1 having v_1 as a terminal vertex. Now, let

$P_1 = P \circ (v_1, v_2)$

$P_2 = (v_2, v_3, \ldots, v_r)$ and

$P_3 = (v_r, v_1)$.

If $deg\ v_1$ is even, then $deg_{G_1}\ v_1$ is even. Let $P = (x_1, x_2, \ldots, x_r, v_1, x_{r+1}, \ldots, x_s)$ be a path in ψ_1 having v_1 as an internal vertex. Now, let

$P_1 = (x_1, x_2, \ldots, x_r, v_1, v_2)$

$P_2 = (x_s, x_{s-1}, \ldots, x_{r+1}, v_1, v_r)$ and

$P_3 = (v_2, v_3, \ldots, v_r)$.

Then $\psi = \{\psi_1 - \{P\}\} \cup \{P_1, P_2, P_3\}$ is a simple path cover of G and hence $\pi_s(G) \leq |\psi_1| + 2 = \frac{k}{2} + 2$. Further, for any simple path cover ψ of G, all the k vertices of odd degree and at least two vertices on C are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 2$, so that $\pi_s(G) = q - t \geq \frac{k}{2} + 2$. Thus $\pi_s(G) = \frac{k}{2} + 2$.

Case 3. $m = 2$.

Let v_1 and v_i, where $2 \leq i \leq r$, be the vertices of degree greater than 2 on C. Let P and Q denote respectively the (v_1, v_i)-section and (v_i, v_1)-section of C. Let v_j be an internal vertex of P(say). Let R_1 and R_2 be the (v_1, v_j)-section of P and (v_j, v_i)-section P respectively. Let G_1 be the graph obtained by deleting all the internal vertices of P.

Subcase 3.1 Both $deg\ v_1$ and $deg\ v_i$ are odd.

Then both $deg_{G_1}\ v_1$ and $deg_{G_1}\ v_i$ are even. Hence G_1 is a tree with $k-2$ odd vertices so that $\pi_s(G_1) = \frac{k}{2} - 1$. Let ψ_1 be a minimum simple path cover of G_1 . Then $\psi = \psi_1 \cup \{R_1, R_2\}$ is a simple path cover of G and $|\psi| = \frac{k}{2} + 1$.Hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.2 Both $deg\ v_1$ and $deg\ v_i$ are even.

Then $deg_{G_1}\ v_1$ and $deg_{G_1}\ v_i$ are odd. Hence G_1 is a tree with $k+2$ vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2} + 1$. Let ψ_1 be a minimum simple path cover of G_1.

Suppose v_1 and v_i are terminal vertices of two different paths in ψ_1, say P_1 and P_2 respectively. Now, let

$Q_1 = P_1 \circ R_1$

$Q_2 = P_2 \circ R_2^{-1}$ and

$\psi\ = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2\}$.

Suppose there exists a path P_1 in ψ_1 having both v_1 and v_i as its end vertices. Then let $P_1 = Q$. Let P_2 be an u_1-w_1 path in ψ_1 having v_1 as an internal vertex and P_3 be an u_2-w_2 path in ψ_1 having v_i as an internal vertex. Let S_1 and S_2 be the (u_1, v_1)-section of P_2 and (w_1, v_1)-section of P_2 respectively. Let S_3 and S_4 be the (u_2, v_i)-section of P_3 and (w_2, v_i)-section of P_3 respectively. Now, let

$Q_1 = S_1 \circ P_1 \circ S_3^{-1}$

$Q_2 = S_2 \circ R_1$

$Q_3 = S_4 \circ R_2^{-1}$ and

$\psi\ = \{\psi_1 - \{P_1, P_2, P_3\}\} \cup \{Q_1, Q_2, Q_3\}$.

Then ψ is a simple path cover of G and $|\psi| = |\psi_1| = \frac{k}{2} + 1$ and hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Subcase 3.3 $deg\ v_1$ is odd and $deg\ v_i$ is even.

Then $deg_{G_1}\ v_1$ is even and $deg_{G_1}\ v_i$ is odd. Hence G_1 is a tree with k vertices of odd degree so that $\pi_s(G_1) = \frac{k}{2}$. Let ψ_1 be a minimum simple path cover of G_1. Let P_1 be the path in ψ_1 having v_i as a terminal vertex.

If $E(P_1) \cap E(Q) = \phi$, let

$Q_1 = P_1 \circ R_2^{-1}$

$Q_2 = R_1$ and

$\psi\ = \{\psi_1 - \{P_1\}\} \cup \{Q_1, Q_2\}$.

Suppose $E(P_1) \cap E(Q) \neq \phi$. Since $deg_{G_1}\ v_i \geq 3$, there exists an u_1-w_1 path in ψ_1, say P_2, having v_i as an internal vertex. Let S_1 and S_2 be the (w_1, v_i)-section of P_2 and (u_1, v_i)-section of P_2 respectively. Now, let

$Q_1 = P_1 \circ S_1^{-1}$

$Q_2 = S_2 \circ R_2^{-1}$

$Q_3 = R_1$ and
$\psi = \{\psi_1 - \{P_1, P_2\}\} \cup \{Q_1, Q_2, Q_3\}$.
Then ψ is a simple path cover of G and $|\psi| = |\psi_1| + 1 = \frac{k}{2} + 1$. Hence $\pi_s(G) \leq \frac{k}{2} + 1$.

Thus in either of the above subcases, we have $\pi_s(G) \leq \frac{k}{2} + 1$. Also, for any simple path cover ψ of G all the k vertices of odd degree and at least one vertex on C are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 1$, so that $\pi_s(G) = q - t \geq \frac{k}{2} + 1$.

Hence $\pi_s(G) = \frac{k}{2} + 1$.

Case 4. $m \geq 3$.

Let $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$, where $1 \leq i_1 < i_2 < \cdots < i_s \leq r$ and $s \geq 3$, be the vertices of degree greater than 2 on C. Let ψ_{i_j}, $1 \leq j \leq s$, be a minimum simple path cover of the tree rooted at v_{i_j}. Consider the vertices v_{i_1}, v_{i_2} and v_{i_3}. For each j, where $1 \leq j \leq 3$, let P_j be the path in ψ_{i_j} in which v_{i_j} is a terminal vertex if *deg* v_{i_j} is odd, otherwise let P_j be an u_j-w_j path in ψ_{i_j} in which v_{i_j} is an internal vertex and R_j and S_j be the (u_j, v_{i_j}) and (w_j, v_{i_j})-sections of P_j respectively. Further, let $P = (v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$, $Q = (v_{i_2}, v_{i_2+1}, \ldots, v_{i_3})$ and $R = (v_{i_3}, v_{i_3+1}, \ldots, v_{i_1})$.

If *deg* v_{i_1}, *deg* v_{i_2} and *deg* v_{i_3} are even, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ R_3^{-1}$ and $Q_3 = S_3 \circ R \circ S_1^{-1}$.

If *deg* v_{i_1}, *deg* v_{i_2} and *deg* v_{i_3} are odd, let $Q_1 = P_1 \circ P$, $Q_2 = P_2 \circ Q$ and $Q_3 = P_3 \circ R$.

If *deg* v_{i_1}, *deg* v_{i_2} are odd and *deg* v_{i_3} is even, let $Q_1 = P_1 \circ P \circ P_2^{-1}$, $Q_2 = R_3 \circ Q^{-1}$ and $Q_3 = S_3 \circ R$.

If *deg* v_{i_1}, *deg* v_{i_2} are even and *deg* v_{i_3} is odd, let $Q_1 = R_1 \circ P \circ R_2^{-1}$, $Q_2 = S_2 \circ Q \circ P_3^{-1}$ and $Q_3 = R \circ S_1^{-1}$.

Then $\psi = (\bigcup_{j=1}^{s} \psi_{i_j} - \{P_1, P_2, P_3\}) \cup \{Q_1, Q_2, Q_3\}$ is a simple path cover of G such that every vertex of odd degree is an external vertex of exactly one path in ψ and no vertex of even degree is an external vertex of any path in ψ. Hence $\pi_s(G) = \frac{k}{2}$. □

Corollary 2.10 *Let G be as in Theorem 2.9. Then $\pi_s(G) = \pi(G)$ if and only if $m \geq 3$.*

Proof The proof follows from Theorem 2.9 and Theorem 1.4. □

We observe that there are infinite families of graphs such as trees and unicyclic graphs having at least three vertices of degree greater than 2 on C for which $\pi_s = \pi$ and so the following problem arises naturally.

Problem 2.11 *Characterize graphs for which $\pi_s = \pi$.*

Theorem 2.12 *For a wheel $W_n = K_1 + C_{n-1}$, we have*

$$\pi_s(W_n) = \begin{cases} 6 & \text{if } n = 4 \\ \left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{if } n \geq 5 \end{cases}$$

Proof Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0 v_i : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1 v_{n-1}\}$.

If $n = 4$, then $W_n = K_4$ and hence $\pi_s(W_n) = 6$.
Now, suppose $n \geq 5$. Let $r = \left\lfloor \frac{n}{2} \right\rfloor$
If n is odd, let

$P_i = (v_i, v_0, v_{r+i})$, $i = 1, 2, \ldots, r$.
$P_{r+1} = (v_1, v_2, \ldots, v_r)$,
$P_{r+2} = (v_1, v_{2r}, v_{2r-1}, \ldots, v_{r+2})$ and
$P_{r+3} = (v_r, v_{r+1}, v_{r+2})$.

If n is even, let

$P_i = (v_i, v_0, v_{r-1+i})$, $i = 1, 2, \ldots, r-1$.
$P_r = (v_0, v_{2r-1})$,
$P_{r+1} = (v_1, v_2, \ldots, v_{r-1})$,
$P_{r+2} = (v_1, v_{2r-1}, \ldots, v_{r+1})$ and
$P_{r+3} = (v_{r-1}, v_r, v_{r+1})$.

Then $\psi = \{P_1, P_2, \ldots, P_{r+3}\}$ is a simple path cover of W_n. Hence $\pi_s(W_n) \leq r+3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Further, for any simple path cover ψ of W_n at least three vertices on $C = (v_1, v_2, \ldots, v_{n-1})$ are terminal vertices of paths in ψ. Hence $t \leq q - \frac{k}{2} - 3$, so that $\pi_s(W_n) = q - t \geq \frac{k}{2} + 3 = \left\lfloor \frac{n}{2} \right\rfloor + 3$. Thus $\pi_s(W_n) = \left\lfloor \frac{n}{2} \right\rfloor + 3$. □

Remark 2.13 Since every simple acyclic graphoidal cover of a graph G is a simple path cover of G and every simple path cover of G is a path cover of G, we have $\eta_{as} \geq \pi_s \geq \pi$. These parameters may be either equal or all distinct as shown below. For the graph G_1 given in Figure 2, $\eta_{as}(G_1) = 7, \pi_s(G_1) = 6, \pi(G_1) = 5$ and for the graph G_2 given in Fig.2.2, we have $\eta_{as}(G_2) = \pi_s(G_2) = \pi(G_2) = 3$.

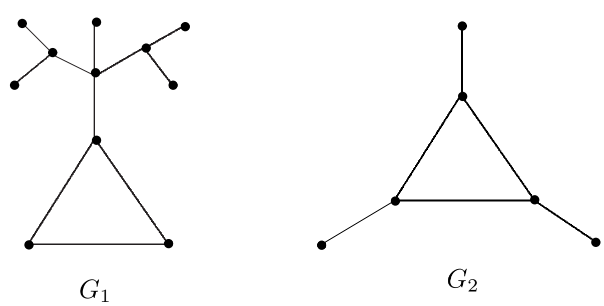


Fig.2.2

Problem 2.14 *Characterize graphs for which $\eta_{as} = \pi_s = \pi$.*

We now proceed to obtain some bounds for π_s.

Theorem 2.15 *For any graph G, $\pi_s(G) \geq \left\lceil \frac{\Delta}{2} \right\rceil$. Further, the following are equivalent.*

(i) $\pi_s(G) = \left\lceil \frac{\Delta}{2} \right\rceil$.
(ii) $\eta_{as}(G) = \Delta - 1$.
(iii) G is homeomorphic to a star.

Proof Since $\pi_s \geq \pi$, the inequality follows from Theorem 1.5.

Suppose $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Let $\psi = \{P_1, P_2, \ldots, P_r\}$, where $r = \lceil \frac{\Delta}{2} \rceil$ be a minimum simple path cover of G. Let v be a vertex of G with $deg\ v = \Delta$. Then v lies on each P_i and v is an internal vertex of all the paths in ψ except possibly for at most one path. Hence $V(P_i) \cap V(P_j) = \{v\}$, for all $i \neq j$, so that G is homeomorphic to a star. Obviously, if G is homeomorphic to a star, then $\pi_s(G) = \lceil \frac{\Delta}{2} \rceil$. Thus (i) and (iii) are equivalent. Similarly the equivalence of (ii) and (iii) can be proved. □

Theorem 2.16 *For any graph G, $\pi_s(G) \geq \binom{\omega}{2}$, where ω is the clique number of G.*

Proof Let H be a maximum clique in G so that $|E(H)| = \binom{\omega}{2}$. Let ψ be a simple path cover of G. Since any path in ψ covers at most one edge of H, it follows that $\pi_s(G) \geq \binom{\omega}{2}$. □

In the following theorem we characterize cubic graphs for which $\pi_s = \binom{\omega}{2}$.

Theorem 2.17 *Let G be a cubic graph. Then $\pi_s(G) = \binom{\omega}{2}$ if and only if $G = K_4$.*

Proof Let G be a cubic graph with $\pi_s(G) = \binom{\omega}{2}$. Clearly $\omega = 3$ or 4. Suppose $\omega = 3$. Then it follows from Corollary 2.6 that $\pi_s(G) \geq \frac{p}{2}$ so that $p = 6$. Hence G is isomorphic to the cartesian product $K_3 \times K_2$ and it can be shown that $\pi_s(K_3 \times K_2) = 6 \neq \binom{\omega}{2}$. Thus $\omega = 4$ and consequently $G = K_4$. □

Problem 2.18 *Characterize graphs for which $\pi_s(G) = \binom{\omega}{2}$.*

If $\Delta \leq 3$, then every simple path cover of G is a simple acyclic graphoidal cover of G and hence $\eta_{as}(G) = \pi_s(G)$. However, the converse is not true. For the complete graph $K_p (p \geq 5)$, $\pi_s = \eta_{as}$ whereas $\Delta \geq 4$. We now prove that the converse is true for trees and unicyclic graphs.

Theorem 2.19 *Let G be a tree. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.*

Proof Let G be a tree with $\eta_{as}(G) = \pi_s(G)$.

Suppose $\Delta \geq 4$. Let v be a vertex of G with $deg\ v \geq 4$.

Let ψ be a minimum simple acyclic graphoidal cover of G. Let P_1 and P_2 be two paths in ψ having v as a terminal vertex. Let $Q = P_1 \circ P_2^{-1}$. Since G is a tree, Q is an induced path and hence $\psi_1 = (\psi - \{P_1, P_2\}) \cup \{Q\}$ is a simple path cover of G with $|\psi_1| = |\psi| - 1 = \eta_{as} - 1$ so that $\pi_s(G) \leq \eta_{as}(G) - 1$, which is a contradiction. Hence $\Delta \leq 3$. □

Theorem 2.20 *Let G be a unicyclic graph. Then $\eta_{as}(G) = \pi_s(G)$ if and only if $\Delta \leq 3$.*

Proof Let G be a unicyclic graph with $\eta_{as}(G) = \pi_s(G)$. Let k denote the number of vertices of odd degree and n be the number of pendant vertices of G.

It follows from Theorem 1.10 and Theorem 2.9 that $k = 2n$. Now, suppose $\Delta > 3$. Then

$$\begin{aligned}
2q &= \sum_{\substack{v \in V(G) \\ deg\ v = 1}} deg\ v + \sum_{\substack{v \in V(G) \\ deg\ v > 1 \\ \text{and is odd}}} deg\ v + \sum_{\substack{v \in V(G) \\ deg\ v > 1 \\ \text{and is even}}} deg\ v \\
&> n + 3(k - n) + 2(p - k) \\
&= 2p,
\end{aligned}$$

which is a contradiction. Hence $\Delta \leq 3$. □

The above results lead to the following problem.

Problem 2.21 *Characterize graphs for which $\eta_{as}(G) = \pi_s(G)$.*

In the following theorem we establish a relation connecting the parameters η_{as} and π_s.

Theorem 2.22 *For any (p, q)-graph G, $\eta_{as}(G) \leq \pi_s(G)+q-p+n-\frac{k}{2}$, where n and k respectively denote the number of pendant vertices and the number of odd vertices of G. Further, equality holds if and only if $\pi_s(G) = \frac{k}{2}$.*

Proof Let ψ be a minimum simple path cover of G. Let $i(v)$ denote the number of paths in ψ having $v \in V$ as an internal vertex. If $i(v) \geq 2$, let P_i, where $1 \leq i \leq i(v)$, be the u_i-w_i path in ψ having v as an internal vertex and let Q_i and R_i, where $2 \leq i \leq i(v)$, respectively denote the (v, w_i)-section and (v, u_i)-section of P_i. Let ψ_1 be the collection of paths obtained from ψ by replacing $P_2, P_3, \ldots, P_{i(v)}$ by $Q_2, Q_3, \ldots, Q_{i(v)}$ and $R_2, R_3, \ldots, R_{i(v)}$ for every $v \in V$ with $i(v) \geq 2$. Then ψ_1 is a simple acyclic graphoidal cover of G with $|\psi_1| = \pi_s(G) + \sum_{\substack{v \in V \\ i(v) \geq 2}} (i(v)-1)$.

Since $i(v) \leq \left\lfloor \frac{deg\ v}{2} \right\rfloor$, it follows that

$$\begin{aligned}
\eta_{as}(G) &\leq \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 4}} \left(\left\lfloor \frac{deg\ v}{2} \right\rfloor - 1 \right) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 2}} \left(\left\lfloor \frac{deg\ v}{2} \right\rfloor - 1 \right) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 2}} \left\lfloor \frac{deg\ v}{2} \right\rfloor - (p-n) \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 2 \\ \text{and is odd}}} \frac{deg\ v-1}{2} + \sum_{\substack{v \in V \\ deg\ v \geq 2 \\ \text{and is even}}} \frac{deg\ v}{2} - p + n \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 2 \\ \text{and is odd}}} \frac{deg\ v}{2} - \frac{k-n}{2} + \sum_{\substack{v \in V \\ deg\ v \geq 2 \\ \text{and is even}}} \frac{deg\ v}{2} - p + n \\
&= \pi_s(G) + \sum_{\substack{v \in V \\ deg\ v \geq 2}} \frac{deg\ v}{2} - \frac{k}{2} + \frac{n}{2} - p + n \\
&= \pi_s(G) + \sum_{v \in V} \frac{deg\ v}{2} - \frac{k}{2} - p + n. \\
&= \pi_s(G) + q - p + n - \frac{k}{2}.
\end{aligned}$$

Thus $\eta_{as}(G) \leq \pi_s(G)+q-p+n-\frac{k}{2}$. Further, it is clear that $\eta_{as}(G) = \pi_s(G)+q-p+n-\frac{k}{2}$ if and only if there exist a minimum simple path cover ψ of G such that $i(v) = \left\lfloor \frac{deg\ v}{2} \right\rfloor$ for all $v \in V$. Hence it follows from Corollary 2.6 that $\eta_{as}(G) = \pi_s(G) + q - p + n - \frac{k}{2}$ if and only if $\pi_s = \frac{k}{2}$. □

Corollary 2.23 *If $\pi_s(G) = \frac{k}{2}$, then $\eta_{as}(G) = q - p + n$.*

Proof Suppose $\pi_s(G) = \frac{k}{2}$. By Theorem 2.22, we have $\eta_{as}(G) \leq q-p+n$. Hence it follows from Theorem 1.9 that $\eta_{as}(G) = q-p+n$. $\square$

Remark 2.24 The converse of Corollary 2.23 is not true. For example, $\eta_{as}(K_{4,5}) = q-p = 11$, whereas $\pi_s(K_{4,5}) > 2 = \frac{k}{2}$.

References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. pure appl. Math.,* **18**(10)(1987), 882 - 890.

[2] S. Arumugam, B. D. Acharya and E. Sampathkumar, Graphoidal covers of a graph - A creative review, *Proceedings of the National workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Eds. S. Arumugam, B. D. Acharya and E. Sampathkumar, Tata McGraw Hill, (1996), 1 - 28.

[3] S. Arumugam and I. Sahul Hamid, Simple Acyclic Graphoidal covers in a graph, *Australasian Journal of Combinatorics*, **37**(2007), 243 -255.

[4] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Math.,* **190**(1998),67 - 77.

[5] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass, 1972.

[6] F. Harary, Covering and packing in graphs I, *Ann. N. Y. Acad. Sci.,* **175**(1970), 198 - 205.

[7] F. Harary and A. J. Schwenk, Evolution of the path number of a graph, covering and packing in graphs II, *Graph Theory and Computing*, Ed. R. C. Road, Academic Press, New York, (1972), 39 - 45.

[8] L.F. Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005

[9] B. Peroche, The path number of some multipartite graphs, *Annals of Discrete Math.,* **9**(1982), 193 - 197.

[10] R. G. Stanton, D. D. Cowan and L. O. James, Some results on path numbers, *Proc. Louisiana Conf. on Combinatorics, Graph Theory and computing* (1970), 112 - 135.

[11] R. G. Stanton, D. D. Cowan and L. O. James, Tripartite path number, *Graph Theory and Computing,* Eds. R. C. Read, Academic Press, New York, (1973), 285 - 294.

Math.Combin.Book.Ser. Vol.3 (2008), 105-110

On the AVSDT-Coloring of $S_m + W_n$

Zhongfu Zhang[1], Enqiang Zhu[1], Baogen Xu[2], Yuhong Zhang[1], Ji Zhang[1] and Jingwen Li[3]

1. Institute of Applied Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P.R. China
2. Department of mathametics, East China Jiaotong University, Nanchang, 330013, P.R.China
3. School of Information and Electronic, Lanzhou Jiaotong University, Lanzhou 730070, P.R. China

E-mail: zhang_zhong_fu@yahoo.com.cn

Abstract: For any vertex $u \in V(G)$, let $T_N(u) = \{u\} \cup \{uv|uv \in E(G), v \in V(G)\} \cup \{v \in V(G)|uv \in E(G)\}$ and f a total $k-$coloring on G. The total-color neighbor of a vertex u of G is the color set $C_f(x) = \{f(x)|x \in T_N(u)\}$. For any adjacent vertices x and y of $V(G)$ such that $C_f(x) \neq C_f(y)$, we refer to f as a k *AVSDT-coloring of* G (the abbreviation of *adjacent-vertex-strongly-distinguishing total coloring of* G). The AVSDT-coloring number of G, denoted by $\chi_{ast}(G)$ is the minimal number of colors required for an AVSDT-coloring of G. A Smarandachely total coloring of a graph G is an AVSDT-coloring with $|C_f(x) \backslash C_f(y)| \geq 2$ and $|C_f(y) \backslash C_f(x)| \geq 2$. In this paper, we calculate the AVSDT-coloring number of $S_m + W_n$.

Keywords: Smarandachely total coloring, join graph, AVSDT-coloring number.

AMS(2000): 05C15, 94C15.

§1. Introduction

Graph coloring is a very important part of graph theory[1]. Recently, the central part is to get the chromatic number of the relate coloring. While it is a NP-problem to get the relate chromatic number for most graphs. Now these vertex distinguishing edge coloring[2], adjacent-strongly edge coloring[3], $D(\beta)$-vertex distinguishing edge coloring[4], adjacent-vertex-distinguishing total coloring[5], $D(\beta)$-vertex distinguishing total coloring[6], vertex-distinguishing total coloring[7], adjacent-vertex-strongly-distinguishing total coloring[8], and a Smarandachely total coloring etc. for a graph, are becoming an interesting research objects for researchers coming from information or computer sciences.

Definition 1.1[8] *Let $G(V, E)$ be a simple connected graph with $|V(G)| \geq 3$, k is a positive integer, f is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$. If f satisfies*

(1) *for any $uv \in E(G)$, we have $f(u) \neq f(v), f(u) \neq f(uv) \neq f(v)$;*

(2) *for any adjacent edges $uv, uw \in E(G)(v \neq w)$, we have$f(uv) \neq f(uw)$;*

(3) *for any edge $uv \in V(G)$, we have $C(u) \neq C(v)$,*

where $C(u) = \{f(u)\} \cup \{f(v)|uv \in E(G)\} \cup \{f(uv)|uv \in E(G)\}$, then f is called adjacent-vertex-strongly-distinguishing total coloring of G, denoted by k-AVSDTC of G for short, and

[1]Supported by NNSFC (10771091 and 10661007).

[2]Received July 8, 2008. Accepted September 26, 2008.

the number

$$\chi_{ast}(G) = min\{k|k - AVSDTC \quad of \quad G\}$$

is called the AVSDT-number of G.

Definition 1.2[1] *For a graph G and H with $V(G)\cap V(H) = \emptyset, E(G)\cap E(H) = \emptyset$, a new graph $G + H$ called the join of G and H is constructed by*

$$V(G + H) = V(G) \cup V(H), E(G + H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}.$$

We once presented a conjecture on the AVSDT-number of a simple connected graph in [8] following.

Conjecture *Let $G(V, E)$ be a simple connected graph with $V(G) \geq 3$. Then $\chi_{ast}(G) \leq n + \lceil\log_2^n\rceil + 1$, where $\lceil\log_2^n\rceil$ denotes the minimal integer not less than log_2^n, and the equality holds if and only if G is a complete graph K_n with $n = 2^k - 2$.*

For terminologies and notations not defined here, we refer to the reference [1].

§2. Main results

Lemma 2.1 [8] *For a simple graph with no isolated edge, if $uv \in E(G)$ and $d(u) = d(v) = \Delta(G)$, then*

$$\chi_{ast}(G) \geq \Delta(G) + 2.$$

Remark When min{m,n}≤4, the AVSDTC-number of $S_m + W_n$ can be obtained easily.

Theorem 2.2 *Suppose $min\{m, n\} \geq 5$, then*

$$\chi_{ast}(S_m + W_n) = m + n + 3.$$

Proof We can know that if $\Delta(u_0) = \Delta(v_0) = m+n+1$, then $\chi_{ast}(S_m+W_n) \geq m+n+3$ by Lemma 2.1. In order to prove $\chi_{ast}(S_m + W_n) = m + n + 3$, we only need to give a $(m + n + 3) - AVSDTC - coloring\ of\ S_m + W_n$. Suppose $V(S_m) = \{u_i|i = 0, 1, \cdots, m\}$, $E(S_m) = \{u_0u_i|i = 1, 2, \cdots, m\}, V(W_n) = \{v_i|i = 0, 1, \cdots, n\}, E(W_n) = \{v_0v_i|i = 1, 2, \cdots, n\} \bigcup\{v_1v_2, v_2v_3, \cdots, v_nv_1\}$.

Our discussion is divided into two cases by constructing the mapping $f : V(G) \cup E(G) \to \{1, 2, \cdots, m + n + 3\}$.

Case 1. $m \geq n \geq 5$

In this case, define

$f(u_0) = 1; f(u_i) = 2, i = 1, 2, \cdots, m; f(v_j) = j + 3, j = 0, 1, \cdots, n;$

$f(u_0v_j)=j+2, j=0,1,\cdots,n-2; f(u_0v_{n-1})=m+n+1; f(u_0v_n)=m+n+2;$

$f(u_0u_i)=n+2+i, i=1,2,\cdots,m-2; f(u_0u_{m-1})=n+1; f(u_0u_m)=n+2;$

$f(u_iv_j)=i+j+3, i=1,2,\cdots,m-1, j=0,1,2,\cdots,n-2;$

$f(u_iv_j)=i+j+4, i=1,2,\cdots,m-4, j=n-1,n;$

$f(u_{m-3}v_{n-1})=m+n; f(u_{m-3}v_n)=3; f(u_{m-2}v_{n-1})=3; f(u_{m-2}v_n)=4;$

$f(u_{m-1}v_{n-1})=4; f(u_{m-1}v_n)=5; f(u_mv_j)=m+j+3, j=0,1,\cdots,n-3;$

$f(u_mv_{n-2})=4; f(u_mv_{n-1})=5; f(u_mv_n)=6;$

$f(v_0v_j)=m+j+4, j=1,2,\cdots,n-1; f(v_0v_n)=1; f(v_1v_n)=m+n+3.$

For $j=1,2,\cdots,n-1$, if $n\equiv 0(mod2)$, then let $f(v_jv_{j+1})=\begin{cases}2, & j\equiv 1(mod2)\\ 1, & j\equiv 0(mod2)\end{cases}$ and if $n\equiv 1(mod2)$, then let $f(v_jv_{j+1})=\begin{cases}1, & j\equiv 1(mod2);\\ 2, & j\equiv 0(mod2).\end{cases}$

We can find classes C by the construction f following.

$C(u_0)=\{1,2,\cdots,m+n+2\};$

$C(u_i)=\{1,2,\cdots,n+i+4\}, i=1,2,\cdots,m-5;$

$C(u_i)=\{1,2,\cdots,m+n\}, i=m-4,m-3,\cdots,m;$

$C(v_0)=\{1,2,\cdots,m+3,m+5,m+6,\cdots,m+n+3\}; C(v_1)=\{1,2,\cdots,m+5,m+n+3\};$

$C(v_i)=\{1,2,3,i+2,i+3,\cdots,m+i+2,m+i+3,m+i+4\}, i=2,3,\cdots,n-3;$

$C(v_{n-2})=\{1,2,3,4,n,n+1,\cdots,m+n,m+n+2\};$

$C(v_{n-1})=\{1,2,3,4,5,n+1,n+2,\cdots,m+n-1,m+n,m+n+1,m+n+3\};$

$C(v_n)=\{1,2,3,4,5,6,n+2,n+3,n+5,n+6,\cdots,m+n,m+n+2,m+n+3\}.$

Obviously, $C(u)\neq C(v)$ for all the vertices in S_m+W_n for $\forall uv\in E(S_m+W_n)$. So $\chi_{ast}(S_m+W_n)=m+n+3$.

Case 2. $n>m\geq 5$

Define f as follows in this case.

$f(u_0)=1; f(u_i)=2, i=1,2,\cdots,m; f(v_j)=j+3, j=0,1,\cdots,n;$

$f(u_0v_j)=j+2, j=0,1,\cdots,n-2; f(u_0v_{n-1})=m+n+1; f(u_0v_n)=m+n+2;$

$f(u_0u_i)=n+2+i, i=1,2,\cdots,m-2; f(u_0u_{m-1})=n+1; f(u_0u_m)=n+2;$

$f(u_iv_j)=i+j+3, i=1,2,\cdots,m-2, j=1,2,\cdots,n-2;$

$f(u_iv_j)=i+j+4, i=1,2,\cdots,m-4, j=n-1,n;$

$f(u_{m-3}v_{n-1})=m+n; f(u_{m-3}v_n)=3; f(u_{m-2}v_{n-1})=3; f(u_{m-2}v_n)=4;$

$f(u_{m-1}v_j)=n+2+j, j=0,1,\cdots,m-2;\ f(u_{m-1}v_j)=5+j-m, j=m-1,m,\cdots,n;$

$f(u_mv_j)=n+3+j, j=0,1,\cdots,m-3;\ f(u_mv_j)=6+j-m, j=m-2,m,\cdots,n.$

For $j = 1, 2, \cdots, n-1$, if $n \equiv 0(mod2)$, let $f(v_jv_{j+1}) = \begin{cases} 2, & j \equiv 1(mod2) \\ 1, & j \equiv 0(mod2) \end{cases}$ and if $n \equiv 1(mod2)$, then $f(v_jv_{j+1}) = \begin{cases} 1, & j \equiv 1(mod2); \\ 2, & j \equiv 0(mod2). \end{cases}$

When $n - m = 1$, define

$f(v_0v_j) = m + j + 4, j = 1, 2, \cdots, n-2;$

$f(v_0v_{n-1}) = n + 1; f(v_0v_n) = 1; f(v_1v_n) = m + n + 3.$

When $n - m \geq 2$, define

$f(v_0v_j) = m + j + 2, j = 1, 2, \cdots, n-m-1; f(v_0v_{n-m}) = m + n + 1;$

$f(v_0v_{n-m+1}) = m + n + 2; f(v_0v_j) = m + j + 2, j = n-m+2, n-m+3, \cdots, n-2;$

$f(v_0v_{n-1}) = m + n + 3; f(v_0v_n) = 1; f(v_1v_n) = m + n + 3,$

Then we know these classes C following by the definition of f.

$C(u_0) = \{1, 2, \cdots, m+n+2\};$

$C(u_i) = \{1, 2, \cdots, n+i+4\}, i = 1, 2, \cdots, m-5;$

$C(u_i) = \{1, 2, \cdots, m+n\}, i = m-4, m-3, \cdots, m;$

$C(v_0) = \{1, 2, \cdots, m+n+3\}.$

In the case of $n - m = 1$, we find

$C(v_1) = \{1, 2, \cdots, m+2, n+3, n+4, n+5, m+n+3\};$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, n+j+2, n+j+3, n+4+j\}, i = 2, 3, \cdots, n-4;$

$C(v_{n-3}) = \{1, 2, 3, 4, n-1, n-2, \cdots, m+n-2, 2n-1, 2n+1\};$

$C(v_{n-2}) = \{1, 2, 3, 4, 5, n, n+1, \cdots, m+n-1, m+n+3\};$

$C(v_{n-1}) = \{1, 2, 3, , 5, 6, n+1, n+2, \cdots, m+n-1, m+n, m+n+1\};$

$C(v_n) = \{1, 2, 3, 4, 6, 7, n+2, n+3, n+5, n+6, \cdots, m+n, m+n+2, m+n+3\}.$

In the case of $n - m \geq 2$, we have

$C(v_1) = \{1, 2, \cdots, m+3, n+3, n+4, m+n+3\}.$

If n-m-1>m-3, then

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+2, n+j+2, n+j+3\}, j = 2, 3, \cdots, m-3;$

$C(v_{m-2}) = \{1, 2, 3, 4, m, m+1, \cdots, 2m-1, m+n, 2m\};$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = m-1, m, \cdots, n-m-1;$

$C(v_{n-m}) = \{1, 2, 3, n-m+2, n-m+3, \cdots, n+1, 5+n-2m, 6+n-2m, m+n+1\};$

$C(v_{n-m+1}) = \{1, 2, 3, n-m+3, n-m+4, \cdots, n+2, 6+n-2m, 7+n-2m, m+n+2\};$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = n-m+2, \cdots, n-2;$

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\};$

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}.$

If n-m-1<m-3, then we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+2, n+j+2, n+j+3\}, j = 2, 3, \cdots, n-m-1;$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+1\}, j = n-m;$

and if n-m+1<m-2, we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+2\}, j = n-m+1;$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, n+2+j, n+3+j, m+j+2\}, j = n-m+2, \cdots, m-3;$

$C(v_{m-2}) = \{1, 2, 3, 4, m, m+1, \cdots, 2m-1, m+n, 2m\}.$

Now if $n-m+1 = m-2$, we know

$C(v_{m-2}) = \{1, 2, 3, 4, m, m+1, \cdots, 2m-1, m+n, n+m+2\};$

and for other cases, we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = m-1, m, \cdots, n-2;$

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\};$

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}.$

Finally, if $n-m-1 = m-3$, we obtain

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+2, n+j+2, n+j+3\}, j = 2, 3, \cdots, m-3;$

$C(v_j) = \{1, 2, 3, 4, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+1\}, j = n-m;$

$C(v_j) = \{1, 2, 3, 4, 5, j+2, j+3, \cdots, j+m+1, n+3+j, m+n+2\}, j = n-m+1;$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = n-m+2, \cdots, n-2;$

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\};$

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}.$

Obviously, $C(u) \neq C(v)$ for all the vertices in $S_m + W_n$ for $\forall uv \in E(S_m + W_n)$. So $\chi_{ast}(S_m + W_n) = m + n + 3$. □

Acknowledgements

The authors wish to express their appreciation of comments and suggestions from the referees.

References

[1] Bondy J.A. Marty U.S.R, *Graph Theory with applications*, Springer Verlag, 2008.

[2] Burris A. C. and Schelp R. H., Vertex-distinguishing edge-colorings, *J.Graph Theory* 20(2)(1997), 73-82.

[3] Zhang Z. F., Liu L. Z., Wang J. F., Adjacent strong edge coloring of graphs, *Appl. Math. Lett.*,15(2002), 623-626.

[4] Zhang Z.F., Chen X.G., Li J.W. et al, $D(\beta)$-vertex distinguishing edge coloring of graphs, *Acta Mathematica Sinica* ,49(3)(2006), 703-708.

[5] Zhang Z.F., Chen X.G., Li J.W. et al, Adjacent-vertex-distinguishing total coloring of graphs, *Sci. China*, Ser. A, 35(3)(2005), 289-299.

[6] Zhang Z.F., Chen X.G., Li J.W. et al, $D(\beta)$-vertex distinguishing total coloring of graphs, *Acta Mathematica Sinica*,36(10)(2006), 1119-1130.

[7] Zhang Z.F., Qiu P.X., Yao B. et al, Vertex distinguishing total coloring of graphs, *Ars Combin.*, 87(2008), 253-262.

[8] Zhang Z.F., Chen H., Yao B. et al, Adjacent-vertex-strongly-distinguishing edge coloring of graphs, *Sci China*, Ser. A,,51(3)(2008),321-480.

Math.Combin.Book.Ser. Vol.3 (2008), 111-121

Actions of Multi-groups on Finite Sets

Linfan MAO

(Chinese Academy of Mathematics and System Science, Beijing 100080, P.R.China)

E-mail: maolinfan@163.com

Abstract: Classifying objects needs permutation groups in mathematics. Similarly, consideration should be also done for actions of multi-groups, i.e., permutation multi-groups. In this paper, we consider the action of multi-groups on a finite multi-set, its orbits, multi-transitive, primitive, etc. By choosing an element p in or not in a permutation group Γ, define a new operation $\circ_p$ enables us to finding permutation multi-groups. Considering such permutation multi-groups, some interesting results in finite permutation groups are generalized to permutation multi-groups.

Key Words: Action of multi-group, permutation multi-group, representation, transitive.

AMS(2000): 05E25, 20B05, 20B20, 51M15.

§1. Introduction

A bijection $f : X \to X$ is called a *permutation* of X. In the case of finite, there is a useful way for representing a permutation τ on X, $|X| = n$ by a $2 \times n$ table following,

$$\tau = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix},$$

where, $x_i, y_i \in X$ and $x_i \neq x_j$, $y_i \neq y_j$ if $i \neq j$ for $1 \leq i, j \leq n$. For three sets X, Y and Z, let $f : X \to Y$ and $h : Y \to Z$ be mapping. Define a mapping $h \circ f : X \to Z$, called the *composition of f and h* by

$$h \circ f(x) = h(f(x))$$

for $\forall x \in X$. For example, let

$$\tau = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix},$$

and

$$\varsigma = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix}.$$

[1]Received July 8, 2008. Accepted September 28, 2008.

Then

$$\sigma = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n, \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ z_1 & z_2 & \cdots & z_n, \end{pmatrix}.$$

It is well-known that all permutations form a group $\Pi(X)$ under the composition operation. For $\forall p \in \Pi(X)$, define an operation $\circ_p : \Pi(X) \times \Pi(X) \to \Pi(X)$ by

$$\sigma \circ_p \varsigma = \sigma p \varsigma, \quad \text{for } \forall \sigma, \varsigma \in \Pi(X).$$

Then we have

Theorem 1.1 *$(\Pi(X); \circ_p)$ is a group.*

Proof We check these conditions for a group hold in $(\Pi(X); \circ_p)$. In fact, for $\forall \tau, \sigma, \varsigma \in \Pi(X)$,

$$\begin{aligned} (\tau \circ_p \sigma) \circ_p \varsigma &= (\tau p \sigma) \circ_p \varsigma = \tau p \sigma p \varsigma \\ &= \tau p(\sigma \circ_p \varsigma) = \tau \circ_p (\sigma \circ_p \varsigma). \end{aligned}$$

The unit in $(\Pi(X); \circ_p)$ is $1_{\circ_p} = p^{-1}$. In fact, for $\forall \theta \in \Pi(X)$, we have that $p^{-1} \circ_p \theta = \theta \circ_p p^{-1} = \theta$.

For an element $\sigma \in \Pi(X)$, $\sigma^{-1}_{\circ_p} = p^{-1}\sigma^{-1}p^{-1} = (p\sigma p)^{-1}$. In fact,

$$\sigma \circ_p (p\sigma p)^{-1} = \sigma p p^{-1} \sigma^{-1} p^{-1} = p^{-1} = 1_{\circ_p},$$

$$(p\sigma p)^{-1} \circ_p \sigma = p^{-1}\sigma^{-1}p^{-1}p\sigma = p^{-1} = 1_{\circ_p}.$$

By definition, we know that $(\Pi(X); \circ_p)$ is a group. □

Notice that if $p = \mathbf{1}_X$, the operation $\circ_p$ is just the composition operation and $(\Pi(X); \circ_p)$ is the symmetric group $Sym(X)$ on X. Furthermore, Theorem 1.1 opens a general way for constructing multi-groups on permutations, which enables us to find the next result.

Theorem 1.2 *Let Γ be a permutation group on X, i.e., $\Gamma \prec Sim(X)$. For given m permutations $p_1, p_2, \cdots, p_m \in \Gamma$, $(\Gamma; \mathscr{O}_P)$ with $\mathscr{O}_P = \{\circ_p, p \in P\}, P = \{p_i, 1 \le i \le m\}$ is a permutation multi-group, denoted by $\mathscr{G}_X^P$.*

Proof First, we check that $(\Gamma; \{\circ_{p_i}, 1 \le i \le m\})$ is an associative system. Actually, for $\forall \sigma, \varsigma, \tau \in \mathscr{G}$ and $p, q \in \{p_1, p_2, \cdots, p_m\}$, we know that

$$\begin{aligned} (\tau \circ_p \sigma) \circ_q \varsigma &= (\tau p \sigma) \circ_q \varsigma = \tau p \sigma q \varsigma \\ &= \tau p(\sigma \circ_q \varsigma) = \tau \circ_p (\sigma \circ_q \varsigma). \end{aligned}$$

Similar to the proof of Theorem 1.1, we know that $(\Gamma;\circ_{p_i})$ is a group for any integer $i, 1 \leq i \leq m$. In fact, $1_{\circ_{p_i}} = p_i^{-1}$ and $\sigma_{\circ_{p_i}}^{-1} = (p_i \sigma p_i)^{-1}$ in $(\mathscr{G};\circ_{p_i})$. □

§2. Multi-permutation Representations

The construction for permutation multi-groups shown in Theorems $1.1 - 1.2$ can be also transferred to permutations on vector as follows, which is useful in some circumstances.

Choose m permutations $p_1, p_2, \cdots, p_m$ on X. An m-permutation on $x \in X$ is defined by

$$p^{(m)} : x \to (p_1(x), p_2(x), \cdots, p_m(x)),$$

i.e., an m-vector on x.

Denoted by $\Pi^{(s)}(X)$ all such s-vectors $p^{(m)}$. Let $\circ$ be an operation on X. Define a *bullet operation of two m-permutations*

$$P^{(m)} = (p_1, p_2, \cdots, p_m),$$
$$Q^{(sm)} = (q_1, q_2, \cdots, q_m)$$

on $\circ$ by

$$P^{(s)} \bullet Q^{(s)} = (p_1 \circ q_1, p_2 \circ q_2, \cdots, p_m \circ q_m).$$

Whence, if there are l-operations $\circ_i, 1 \leq i \leq l$ on X, we obtain an s-permutation system $\Pi^{(s)}(X)$ under these l bullet operations $\bullet_i, 1 \leq i \leq l$, denoted by $(\Pi^{(s)}(X); \odot_1^l)$, where $\odot_1^l = \{\bullet_i | 1 \leq i \leq l\}$.

Theorem 2.1 *Any s-operation system $(\mathscr{H}, \widetilde{O})$ on $\mathscr{H}$ with units $1_{\circ_i}$ for each operation $\circ_i, 1 \leq i \leq s$ in $\widetilde{O}$ is isomorphic to an s-permutation system $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$.*

Proof For $a \in \mathscr{H}$, define an s-permutation $\sigma_a \in \Pi^{(s)}(\mathscr{H})$ by

$$\sigma_a(x) = (a \circ_1 x, a \circ_2 x, \cdots, a \circ_s x)$$

for $\forall x \in \mathscr{H}$.

Now let $\pi : \mathscr{H} \to \Pi^{(s)}(\mathscr{H})$ be determined by $\pi(a) = \sigma_a^{(s)}$ for $\forall a \in \mathscr{H}$. Since

$$\sigma_a(1_{\circ_i}) = (a \circ_1 1_{\circ_i}, \cdots, a \circ_{i-1} 1_{\circ_i}, a, a \circ_{i+1} 1_{\circ_i}, \cdots, a \circ_s 1_{\circ_i}),$$

we know that for $a, b \in \mathscr{H}$, $\sigma_a \neq \sigma_b$ if $a \neq b$. Hence, π is a $1-1$ and onto mapping. For $\forall i, 1 \leq i \leq s$ and $\forall x \in \mathscr{H}$, we find that

$$\begin{aligned}
\pi(a \circ_i b)(x) &= \sigma_{a \circ_i b}(x) \\
&= (a \circ_i b \circ_1 x, a \circ_i b \circ_2 x, \cdots, a \circ_i b \circ_s x) \\
&= (a \circ_1 x, a \circ_2 x, \cdots, a \circ_s x) \bullet_i (b \circ_1 x, b \circ_2 x, \cdots, b \circ_s x) \\
&= \sigma_a(x) \bullet_i \sigma_b(x) = \pi(a) \bullet_i \pi(b)(x).
\end{aligned}$$

Therefore, $\pi(a \circ_i b) = \pi(a) \bullet_i \pi(b)$, which implies that π is an isomorphism from $(\mathscr{H}, \widetilde{O})$ to $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$. $\square$

According to Theorem 2.1, these algebraic multi-systems are the same as permutation multi-systems, particularly for multi-groups.

Corollary 2.1 *Any s-group $(\mathscr{H}, \widetilde{O})$ with $\widetilde{O} = \{\circ_i | 1 \leq i \leq s\}$ is isomorphic to an s-permutation multi-group $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$.*

Proof It can be shown easily that $(\Pi^{(s)}(\mathscr{H}); \odot_1^s)$ is a multi-group if $(\mathscr{H}, \widetilde{O})$ is a multi-group. $\square$

For the special case of $s = 1$ in Corollary 2.1, we get the well-known Cayley result on groups.

Corollary 2.2(Cayley) *A group G is isomorphic to a permutation group.*

As shown in Theorem 1.2, many operations can be defined on a permutation group G, which enables it to be a permutation multi-group, and generally, these operations $\circ_i, 1 \leq i \leq s$ on permutations in Theorem 2.1 need not to be the composition of permutations. If we choose all $\circ_i, 1 \leq i \leq s$ to be just the composition of permutation, then all bullet operations in $\odot_1^s$ is the same, denoted by $\odot$. We find an interesting result following which also implies the Cayley's result on groups, i.e., Corollary 2.2.

Theorem 2.2 *$(\Pi^{(s)}(\mathscr{H}); \odot)$ is a group of order $\frac{(n!)!}{(n!-s)!}$.*

Proof By definition, we know that

$$P^{(s)}(x) \odot Q^{(s)}(x) = (P_1Q_1(x), P_2Q_2(x), \cdots, P_sQ_s(x))$$

for $\forall P^{(s)}, Q^{(s)} \in \Pi^{(s)}(\mathscr{H})$ and $\forall x \in \mathscr{H}$. Whence, $(1, 1, \cdots, 1)$ (l entries 1) is the unit and $(P^{-(s)}) = (P_1^{-1}, P_2^{-1}, \cdots, P_s^{-1})$ the inverse of $P^{(s)} = (P_1, P_2, \cdots, P_s)$ in $(\Pi^{(s)}(\mathscr{H}); \odot)$. Therefore, $(\Pi^{(s)}(\mathscr{H}); \odot)$ is a group.

Calculation shows that the order of $\Pi^{(s)}(\mathscr{H})$ is

$$\begin{pmatrix} n! \\ s \end{pmatrix} s! = \frac{(n!)!}{(n! - s)!},$$

which completes the proof. $\square$

§3. Action of Multi-group

Let $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ be a multi-group, where $\widetilde{\mathscr{A}} = \bigcup\limits_{i=1}^{m} \mathscr{H}_i$, $\widetilde{\mathscr{O}} = \bigcup\limits_{i=1}^{m} \mathcal{O}_i$, and $\widetilde{X} = \bigcup\limits_{i=1}^{m} X_i$ a multi-set. An *action* φ of $(\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}})$ on $\widetilde{X}$ is defined to be a homomorphism

$$\varphi : (\widetilde{\mathscr{A}}; \widetilde{\mathscr{O}}) \to \bigcup_{i=1}^{m} \mathscr{G}_{X_i}^{P_i}$$

for sets $P_1, P_2, \cdots, P_m \geq 1$ of permutations, i.e., for $\forall h \in \mathscr{H}_i$, $1 \leq i \leq m$, there is a permutation $\varphi(h) : x \to x^h$ with conditions following hold,

$$\varphi(h \circ g) = \varphi(h)\varphi(\circ)\varphi(g), \quad \text{for } h, g \in \mathscr{H}_i \text{ and } \circ \in \mathcal{O}_i.$$

Whence, we only need to consider the action of permutation multi-groups on multi-sets. Let $= (\widetilde{\mathscr{A}; \widetilde{\mathscr{O}}})$ be a multi-group action on a multi-set $\widetilde{X}$, denoted by $\mathscr{G}$. For a subset $\Delta \subset \widetilde{X}$, $x \in \Delta$, we define

$$x^{\mathscr{G}} = \{\ x^g \mid \forall g \in \mathscr{G}\} \ \text{ and } \ \mathscr{G}_x = \{\ g \mid x^g = x, g \in \mathscr{G}\},$$

called the *orbit* and *stabilizer* of x under the action of $\mathscr{G}$ and sets

$$\mathscr{G}_\Delta = \{\ g \mid x^g = x, g \in \mathscr{G} \ \text{ for } \ \forall x \in \Delta\},$$

$$\mathscr{G}_{(\Delta)} = \{\ g \mid \Delta^g = \Delta, g \in \mathscr{G} \ \text{ for } \ \forall x \in \Delta\},$$

respectively. Then we know the result following.

Theorem 3.1 *Let Γ be a permutation group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in \Gamma$ for integers $1 \leq i \leq m$. Then*

(i) $|\mathscr{G}_X^P| = |(\mathscr{G}_X^P)_x||x^{\mathscr{G}_X^P}|$, $\forall x \in X$;

(ii) for $\forall \Delta \subset X$, $((\mathscr{G}_X^P)_\Delta, \mathcal{O}_P)$ is a permutation multi-group if and only if $p_i \in P$ for $1 \leqslant i \leq m$.

Proof By definition, we know that

$$(\mathscr{G}_X^P)_x = \Gamma_x, \ \text{and} \ x^{\mathscr{G}_X^P} = x^\Gamma$$

for $x \in X$ and $\Delta \subset X$. Assume that $x^\Gamma = \{x_1, x_2, \cdots, x_l\}$ with $x^{g_i} = x_i$. Then we must have

$$\Gamma = \bigcup_{i=1}^{l} g_i \Gamma_x.$$

In fact, for $\forall h \in \Gamma$, let $x^h = x_k, 1 \leq k \leq m$. Then $x^h = x^{g_k}$, i.e., $x^{hg_k^{-1}} = x$. Whence, we get that $hg_k^{-1} \in \Gamma_x$, namely, $h \in g_k\Gamma_x$.

For integers $i, j, i \neq j$, there are must be $g_i\Gamma_x \cap g_j\Gamma_x = \emptyset$. Otherwise, there exist $h_1, h_2 \in \Gamma_x$ such that $g_ih_1 = g_jh_2$. We get that $x_i = x^{g_i} = x^{g_jh_2h_1^{-1}} = x^{g_j} = x_j$, a contradiction.

Therefore, we find that

$$|\mathscr{G}_X^P| = |\Gamma| = |\Gamma_x||x^\Gamma| = |(\mathscr{G}_X^P)_x||x^{\mathscr{G}_X^P}|.$$

This is the assertion (i). For (ii), notice that $(\mathscr{G}_X^P)_\Delta = \Gamma_\Delta$ and Γ_Δ is itself a permutation group. Applying Theorem 1.2, we find it. $\square$

Particularly, for a permutation group Γ action on Ω, i.e., all $p_i = \mathbf{1}_X$ for $1 \leq i \leq m$, we get a consequence of Theorem 3.1.

Corollary 3.1 *Let Γ be a permutation group action on Ω. Then*

(*i*) $|\Gamma| = |\Gamma_x||x^\Gamma|$, $\forall x \in \Omega$;

(*ii*) *for* $\forall \Delta \subset \Omega$, Γ_Δ *is a permutation group.*

Theorem 3.2 *Let Γ be a permutation group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$, $p_i \in \Gamma$ for integers $1 \le i \le m$ and $Orb(X)$ the orbital sets of $\mathscr{G}_X^P$ action on X. Then*

$$|Orb(X)| = \frac{1}{|\mathscr{G}_X^P|} \sum_{p \in \mathscr{G}_X^P} |\Phi(p)|,$$

where $\Phi(p)$ is the fixed set of p, i.e., $\Phi(p) = \{x \in X | x^p = x\}$.

Proof Consider a set $E = \{(p, x) \in \mathscr{G}_X^P \times X | x^p = x\}$. Then we know that $E(p, *) = \Phi(p)$ and $E(*, x) = (\mathscr{G}_X^P)_x$. Counting these elements in E, we find that

$$\sum_{p \in \mathscr{G}_X^P} |\Phi(p)| = \sum_{x \in X} (\mathscr{G}_X^P)_x.$$

Now let $x_i, 1 \le i \le |Orb(X)|$ be representations of different orbits in $Orb(X)$. For an element y in $x_i^{\mathscr{G}_X^P}$, let $y = x_i^g$ for an element $g \in \mathscr{G}_X^P$. Now if $h \in (\mathscr{G}_X^P)_y$, i.e., $y^h = y$, then we find that $(x_i^g)^h = x_i^g$. Whence, $x_i^{ghg^{-1}} = x_i$. We obtain that $ghg^{-1} \in (\mathscr{G}_X^P)_{x_i}$, namely, $h \in g^{-1}(\mathscr{G}_X^P)_{x_i} g$. Therefore, $(\mathscr{G}_X^P)_y \subset g^{-1}(\mathscr{G}_X^P)_{x_i} g$. Similarly, we get that $(\mathscr{G}_X^P)_{x_i} \subset g(\mathscr{G}_X^P)_y g^{-1}$, i.e., $(\mathscr{G}_X^P)_y = g^{-1}(\mathscr{G}_X^P)_{x_i} g$. We know that $|(\mathscr{G}_X^P)_y| = |(\mathscr{G}_X^P)_{x_i}|$ for any element in $x_i^{\mathscr{G}_X^P}, 1 \le i \le |Orb(X)|$. This enables us to obtain that

$$\begin{aligned}
\sum_{p \in \mathscr{G}_X^P} |\Phi(p)| &= \sum_{x \in X} (\mathscr{G}_X^P)_x \\
&= \sum_{i=1}^{|Orb(X)|} \sum_{y \in x_i^{\mathscr{G}_X^P}} |(\mathscr{G}_X^P)_{x_i}| \\
&= \sum_{i=1}^{|Orb(X)|} |x_i^{\mathscr{G}_X^P}||(\mathscr{G}_X^P)_{x_i}| \\
&= \sum_{i=1}^{|Orb(X)|} |\mathscr{G}_X^P| = |Orb(X)||\mathscr{G}_X^P|
\end{aligned}$$

by applying Theorem 3.1. This completes the proof. □

For a permutation group Γ action on Ω, i.e., all $p_i = \mathbf{1}_X$ for $1 \le i \le m$, we get the famous *Burnside's Lemma* by Theorem 3.2.

Corollary 3.2(Burnside's Lemma) *Let Γ be a permutation group action on Ω. Then*

$$|Orb(\Omega)| = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} |\Phi(g)|.$$

§4. Transitive Multi-groups

A permutation multi-group $\mathscr{G}_X^P$ is *transitive* on X if $|Orb(X)| = 1$, i.e., for any elements $x, y \in X$, there is an element $g \in \mathscr{G}_X^P$ such that $x^g = y$. In this case, we know formulae following by Theorems 3.1 and 3.2.

$$|\mathscr{G}_X^P| = |X||(\mathscr{G}_X^P)_x| \quad \text{and} \quad |\mathscr{G}_X^P| = \sum_{p \in \mathscr{G}_X^P} |\Phi(p)|$$

Similarly, a permutation multi-group $\mathscr{G}_X^P$ is *k-transitive* on X if for any two k-tuples $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, there is an element $g \in \mathscr{G}_X^P$ such that $x_i^g = y_i$ for any integer $i, 1 \leq i \leq k$. It is well-known that $Sym(X)$ is $|X|$-transitive on a finite set X.

Theorem 4.1 *Let Γ be a transitive group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$ and $p_i \in \Gamma$ for integers $1 \leq i \leq m$. Then for an integer k,*

(i) $(\Gamma; X)$ is k-transitive if and only if $(\Gamma_x; X \setminus \{x\})$ is $(k-1)$-transitive;

(ii) $\mathscr{G}_X^P$ is k-transitive on X if and only if $(\mathscr{G}_X^P)_x$ is $(k-1)$-transitive on $X \setminus \{x\}$.

Proof If Γ is k-transitive on X, it is obvious that Γ is $(k-1)$-transitive on X itself. Conversely, if Γ_x is $(k-1)$-transitive on $X \setminus \{x\}$, then for two k-tuples $(x_1, x_2, \cdots, x_k)$ and $(y_1, y_2, \cdots, y_k)$, there are elements $g_1, g_2 \in \Gamma$ and $h \in \Gamma_x$ such that

$$x_1^{g_1} = x, \quad y_1^{g_2} = x \quad \text{and} \quad (x_i^{g_1})^h = y_i^{g_2}$$

for any integer $i, 2 \leq i \leq k$. Therefore,

$$x_i^{g_1 h g_2^{-1}} = y_i, \quad 1 \leq i \leq k.$$

We know that Γ is 'k-transitive on X. This is the assertion of (i).

By definition, $\mathscr{G}_X^P$ is k-transitive on X if and only if Γ is k-transitive, i.e., $(\mathscr{G}_X^P)_x$ is $(k-1)$-transitive on $X \setminus \{x\}$ by (i), which is the assertion of (ii). $\square$

Applying Theorems 3.1 and 4.1 repeatedly, we get an interesting consequence for k-transitive multi-groups.

Theorem 4.2 *Let $\mathscr{G}_X^P$ be a k-transitive multi-group and $\Delta \subset X$ with $|\Delta| = k$. Then*

$$|\mathscr{G}_X^P| = |X|(|X|-1)\cdots(|X|-k+1|(\mathscr{G}_X^P)_\Delta|.$$

Particularly, a k-transitive multi-group $\mathscr{G}_X^P$ with $|\mathscr{G}_X^P| = |X|(|X|-1)\cdots(|X|-k+1|$ is called a *sharply k-transitive multi-group*. For example, choose $\Gamma = Sym(X)$ with $|X| = n$, i.e., the symmetric group S_n and permutations $p_i \in S_n$, $1 \leq i \leq m$, we get an n-transitive multi-group $(S_n; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}$.

Let Γ be a transitive group action on X and $\mathscr{G}_X^P$ a permutation multi-group $(\Gamma; \mathscr{O}_P)$ with $P = \{p_1, p_2, \cdots, p_m\}, p_i \in \Gamma$ for integers $1 \leq i \leq m$. An equivalent relation R on X is *$\mathscr{G}_X^P$-admissible* if for $\forall (x, y) \in R$, there is $(x^g, y^g) \in R$ for $\forall g \in \mathscr{G}_X^P$. For a given set X and permutation multi-group $\mathscr{G}_X^P$, it can be shown easily by definition that

$$R = X \times X \quad \text{or} \quad R = \{(x,x)|x \in X\}$$

are $\mathscr{G}_X^P$-admissible, called *trivially $\mathscr{G}_X^P$-admissible relations.* A transitive multi-group $\mathscr{G}_X^P$ is *primitive* if there are no $\mathscr{G}_X^P$-admissible relations R on X unless $R = X \times X$ or $R = \{(x,x)|x \in X\}$, i.e., the trivially relations. The next result shows the existence of primitive multi-groups.

Theorem 4.3 *Every k-transitive multi-group $\mathscr{G}_X^P$ is primitive if $k \geq 2$.*

Proof Otherwise, there is a $\mathscr{G}_X^P$-admissible relations R on X such that $R \neq X \times X$ and $R \neq \{(x,x)|x \in X\}$. Whence, there must exists $(x,y) \in R$, $x,y \in X$ and $x \neq y$. By assumption, $\mathscr{G}_X^P$ is k-transitive on X, $k \geq 2$. For $\forall z \in X$, there is an element $g \in \mathscr{G}_X^P$ such that $x^g = x$ and $y^g = z$. This fact implies that $(x,z) \in R$ for $\forall z \in X$ by definition. Notice that R is an equivalence relation on X. For $\forall z_1, z_2 \in X$, we get $(z_1,x),(x,z_2) \in R$. Thereafter, $(z_1,z_2) \in R$, namely, $R = X \times X$, a contradiction. $\Box$

There is a simple criterion for determining which permutation multi-group is primitive by maximal stabilizers following.

Theorem 4.4 *A transitive multi-group $\mathscr{G}_X^P$ is primitive if and only if there is an element $x \in X$ such that $p \in (\mathscr{G}_X^P)_x$ for $\forall p \in P$ and if there is a permutation multi-group $(\mathcal{H};\mathscr{O}_P)$ enabling $((\mathscr{G}_X^P)_x;\mathscr{O}_P) \prec (\mathcal{H};\mathscr{O}_P) \prec \mathscr{G}_X^P$, then $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ or $\mathscr{G}_X^P$.*

Proof If $(\mathcal{H};\mathscr{O}_P)$ be a multi-group with $((\mathscr{G}_X^P)_x;\mathscr{O}_P) \prec (\mathcal{H};\mathscr{O}_P) \prec \mathscr{G}_X^P$ for an element $x \in X$, define a relation

$$R = \{\ (x^g, x^{g\circ h}) \mid g \in \mathscr{G}_X^P,\ h \in \mathcal{H}\ \}.$$

for a chosen operation $\circ \in \mathscr{O}_P$. Then R is a $\mathscr{G}_X^P$-admissible equivalent relation. In fact, it is $\mathscr{G}_X^P$-admissible, reflexive and symmetric by definition. For its transitiveness, let $(s,t) \in R$, $(t,u) \in R$. Then there are elements $g_1, g_2 \in \mathscr{G}_X^P$ and $h_1, h_2 \in \mathcal{H}$ such that

$$s = x^{g_1},\ t = x^{g_1\circ h_1},\ t = x^{g_2},\ u = x^{g_2\circ h_2}.$$

Hence, $x^{g_2^{-1}\circ g_1\circ h_1} = x$, i.e., $g_2^{-1} \circ g_1 \circ h_1 \in \mathcal{H}$. Whence, $g_2^{-1} \circ g_1,\ g_1^{-1} \circ g_2 \in \mathcal{H}$. Let $g^* = g_1$, $h^* = g_1^{-1} \circ g_2 \circ h_2$. We find that $s = x^{g^*}$, $u = x^{g^*\circ h^*}$. Therefore, $(s,u) \in R$. This concludes that R is an equivalent relation.

Now if $\mathscr{G}_X^P$ is primitive, then $R = \{(x,x)|x \in X\}$ or $R = X \times X$ by definition. Assume $R = \{(x,x)|x \in X\}$. Then $s = x^g$ and $t = x^{g\circ h}$ implies that $s = t$ for $\forall g \in \mathscr{G}_X^P$ and $h \in \mathcal{H}$. Particularly, for $g = 1_\circ$, we find that $x^h = x$ for $\forall h \in \mathcal{H}$, i.e., $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P)_x;\mathscr{O}_P)$.

If $R = X \times X$, then $(x, x^f) \in R$ for $\forall f \in \mathscr{G}_X^P$. In this case, there must exist $g \in \mathscr{G}_X^P$ and $h \in \mathcal{H}$ such that $x = x^g, x^f = x^{g\circ h}$. Whence, $g \in ((\mathscr{G}_X^P)_x;\mathscr{O}_P) \prec (\mathcal{H};\mathscr{O}_P)$ and $g^{-1} \circ h^{-1} \circ f \in ((\mathscr{G}_X^P)_x;\mathscr{O}_P) \prec (\mathcal{H};\mathscr{O}_P)$. Therefore, $f \in \mathcal{H}$ and $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P);\mathscr{O}_P)$.

Conversely, assume R to be a $\mathscr{G}_X^P$-admissible equivalent relation and there is an element $x \in X$ such that $p \in (\mathscr{G}_X^P)_x$ for $\forall p \in P$, $((\mathscr{G}_X^P)_x;\mathscr{O}_P) \prec (\mathcal{H};\mathscr{O}_P) \prec \mathscr{G}_X^P$ implies that $(\mathcal{H};\mathscr{O}_P) = ((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ or $((\mathscr{G}_X^P);\mathscr{O}_P)$. Define

$$\mathcal{H} = \{\ h \in \mathscr{G}_X^P \mid (x, x^h) \in R\ \}.$$

Then $(\mathcal{H};\mathscr{O}_P)$ is multi-subgroup of $\mathscr{G}_X^P$ which contains a multi-subgroup $((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ by definition. Whence, $(\mathcal{H};\mathscr{O}_P)=((\mathscr{G}_X^P)_x;\mathscr{O}_P)$ or $\mathscr{G}_X^P$.

If $(\mathcal{H};\mathscr{O}_P)=((\mathscr{G}_X^P)_x;\mathscr{O}_P)$, then x is only equivalent to itself. Since $\mathscr{G}_X^P$ is transitive on X, we know that $R=\{(x,x)|x\in X\}$.

If $(\mathcal{H};\mathscr{O}_P)=\mathscr{G}_X^P$, by the transitiveness of $\mathscr{G}_X^P$ on X again, we find that $R=X\times X$. Combining these discussions, we conclude that $\mathscr{G}_X^P$ is primitive. □

Choose $p=1_X$ for each $p\in P$ in Theorem 4.4, we get a well-known result in classical permutation groups following.

Corollary 4.1 *A transitive group Γ is primitive if and only if there is an element $x\in X$ such that a subgroup H with $\Gamma_x\prec H\prec\Gamma$ hold implies that $H=\Gamma_x$ or Γ.*

§5. Extended Permutation Multi-groups

Let Γ be a permutation group action on a set X and $P\subset\Pi(X)$. We have shown in Theorem 1.2 that $(\Gamma;\mathscr{O}_P)$ is a multi-group if $P\subset\Gamma$. Then *what can we say if not all $p\in P$ are in Γ*? For this case, we introduce a new multi-group $(\widetilde{\Gamma};\mathscr{O}_P)$ on X, the *permutation multi-group generated by P in Γ* by

$$\widetilde{\Gamma}=\{\ g_1\circ_{p_1}g_2\circ_{p_2}\cdots\circ_{p_l}g_{l+1}\mid g_i\in\Gamma,p_j\in P,\ 1\leq i\leq l+1,1\leq j\leq l\ \},$$

denoted by $\left\langle\Gamma_X^P\right\rangle$. This multi-group has good behavior like $\mathscr{G}_X^P$, also a kind way of extending a group to a multi-group. For convenience, a group generated by a set S with the operation in Γ is denoted by $\langle S\rangle_\Gamma$.

Theorem 5.1 *Let Γ be a permutation group action on a set X and $P\subset\Pi(X)$. Then*

(i) $\left\langle\Gamma_X^P\right\rangle=\langle\Gamma\cup P\rangle_\Gamma$, particularly, $\left\langle\Gamma_X^P\right\rangle=\mathscr{G}_X^P$ if and only if $P\subset\Gamma$;

(ii) for any subgroup Λ of Γ, there exists a subset $P\subset\Gamma$ such that

$$\left\langle\Lambda_X^P;\mathscr{O}_P\right\rangle=\left\langle\Gamma_X^P\right\rangle.$$

Proof By definition, for $\forall a,b\in\Gamma$ and $p\in P$, we know that

$$a\circ_p b\ =\ apb.$$

Choosing $a=b=1_\Gamma$, we find that

$$a\circ_p b\ =\ p,$$

i.e., $\Gamma\subset\widetilde{\Gamma}$. Whence,

$$\langle\Gamma\cup P\rangle_\Gamma\subset\left\langle\Gamma_X^P\right\rangle.$$

Now for $\forall g_i\in\Gamma$ and $p_j\in P$, $1\leq i\leq l+1$, $1\leq j\leq l$, we know that

$$g_1 \circ_{p_1} g_2 \circ_{p_2} \cdots \circ_{p_l} g_{l+1} = g_1p_1g_2p_2\cdots p_lg_{l+1},$$

which means that

$$\left\langle \Gamma_X^P \right\rangle \subset \langle \Gamma \cup P\rangle_\Gamma .$$

Therefore,

$$\left\langle \Gamma_X^P \right\rangle = \langle \Gamma \cup P\rangle_\Gamma .$$

Now if $\left\langle \Gamma_X^P \right\rangle = \mathscr{G}_X^P$, i.e., $\langle \Gamma \cup P\rangle_\Gamma = \Gamma$, there must be $P \subset \Gamma$. According to Theorem 1.2, this concludes the assertion (i).

For the assertion (ii), notice that if $P = \Gamma \setminus \Lambda$, we get that

$$\left\langle \Lambda_X^P \right\rangle = \left\langle \Lambda \bigcup P \right\rangle_\Gamma = \Gamma$$

by (i). Whence, there always exists a subset $P \subset \Gamma$ such that

$$\left\langle \Lambda_X^P ; \mathscr{O}_P \right\rangle = \left\langle \Gamma_X^P \right\rangle .$$

$\square$

Theorem 5.2 *Let Γ be a permutation group action on a set X. For an integer $k \geq 1$, there is a set $P \in \Pi(X)$ with $|P| \leq k$ such that $\left\langle \Gamma_X^P \right\rangle$ is k-transitive.*

Proof Notice that the symmetric group $Sym(X)$ is $|X|$-transitive for any finite set X. If Γ is k-transitive on X, choose $P = \emptyset$ enabling the conclusion true. Otherwise, assume these orbits of Γ action on X to be $O_1, O_2, \cdots, O_s$, where $s = |Orb(X)|$. Construct a permutation $p \in \Pi(X)$ by

$$p = (x_1, x_2, \cdots, x_s),$$

where $x_i \in O_i$, $1 \leq i \leq s$ and let $P = \{p\} \subset Sym(X)$. Applying Theorem 5.1, we know that $\left\langle \Gamma_X^P \right\rangle = \langle \Gamma \cup P\rangle_\Gamma$ is transitive on X with $|P| = 1$.

Now for an integer k, if $\left\langle \Gamma_X^{P_1} \right\rangle$ is k-transitive with $|P_1| \leq k$, let $O_1', O_2', \cdots, O_l'$ be these orbits of the stabilizer $\left\langle \Gamma_X^{P_1} \right\rangle_{y_1y_2\cdots y_k}$ action on $X \setminus \{y_1, y_2, \cdots, y_k\}$. Construct a permutation

$$q = (z_1, z_2, \cdots, z_l),$$

where $z_i \in O_i'$, $1 \leq i \leq l$ and let $P_2 = P_1 \cup \{q\}$. Applying Theorem 5.1 again, we find that $\left\langle \Gamma_X^{P_2} \right\rangle_{y_1y_2\cdots y_k}$ is transitive on $X \setminus \{y_1, y_2, \cdots, y_k\}$, where $|P_2| \leq |P_1| + 1$. Therefore, $\left\langle \Gamma_X^{P_2} \right\rangle$ is $(k+1)$-transitive on X with $|P_2| \leq k+1$ by Theorem 2.5.7.

Applying the induction principle, we get the conclusion. $\square$

Notice that any k-transitive multi-group is primitive if $k \geq 2$ by Theorem 4.3. We have a corollary following by Theorem 5.2.

Corollary 5.1 *Let Γ be a permutation group action on a set X. There is a set $P \in \Pi(X)$ such that $\left\langle \Gamma_X^P \right\rangle$ is primitive.*

References

[1] N.L.Biggs and A.T.White, *Permutation Groups and Combinatoric Structure*, Cambridge University Press, 1979.

[2] G.Birkhoff and S.MacLane, *A Survey of Modern Algebra* (4th edition), Macmillan Publishing Co., Inc, 1977.

[3] L.F.Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, American Research Press, 2005.

[4] L.F.Mao, On automorphism groups of maps, surfaces and Smarandache geometries, *Scientia Magna*, Vol.1(2005), No.2,55-73.

[5] L.F.Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[6] L.F.Mao, On algebraic multi-group spaces, *Scientia Magna*, Vol.2,No.1(2006), 64-70.

[7] L.F.Mao, Extending homomorphism theorem to multi-systems, *International J.Math.Combin.*, Vol.3,(2001), 1-27.

[8] F.Smarandache, *A Unifying Field in Logics. Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[9] F.Smarandache, Mixed noneuclidean geometries, *eprint arXiv: math/0010119*, 10/2000.

[10] M.Y.Xu, *Introduction to Group Theory* (in Chinese)(I)(II), Science Press, Beijing, 1999.

If you would not forgotten as soon as you are dead, either write things worth reading or do things worth write.

By Benjamin Franklin, an American Scientist.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in the **Mathematical Combinatorics (International Book Series)** (*ISBN 1-59973-076-6*). An effort is made to publish a paper duly recommended by a referee within a period of 3 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]K. Kawakubo, The Theory of Transformation Groups, Oxford University Press, New York, 1991.

Research papers

[8]K. K. Azad and Gunjan Agrawal, On the projective cover of an orbit space, *J. Austral. Math. Soc.* 46 (1989), 308-312.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be sent as: fair copy on paper, whenever possible scaled to about 200%, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

Reprints: One copy of the journal included his or her paper(s) are provided to the authors freely. Additional sets may be ordered at the time of proof correction.

Mathematical Combinatorics (*International Book Series*), **October, 2008**

Contents

Extending Homomorphism Theorem to Multi-Systems
BY LINFAN MAO ... 01
A Double Cryptography Using the Smarandache Keedwell Cross Inverse Quasigroup
BY T. G. JAÍYÉOLÁ ... 28
On the Time-like Curves of Constant Breadth in Minkowski 3-Space
BY SÜHA YILMAZ AND MELIH TURGUT ... 34
On the Basis Number of the Strong Product of Theta Graphs with Cycles
BY M.M.M. JARADAT, M.F. JANEM AND A.J. ALAWNEH ... 40
Smarandache Curves in Minkowski Space-time
BY MELIH TURGUT AND SÜHA YILMAZ ... 51
The Characterization of Symmetric Primitive Matrices with exponent $n-3$
BY LICHAO, HUANGFU AND JUNLIANG CAI ... 56
The Crossing Number of the Circulant Graph $C(3k-1;\{1,k\})$
BY JING WANG AND YUANQIU HUANG ... 79
On the Edge Geodetic and k**-Edge Geodetic Number of a Graph**
BY A.P. SANTHAKUMARAN AND S.V. ULLAS CHANDRAN ... 85
Simple Path Covers in Graphs
BY S.ARUMUGAM AND I.SAHUL HAMID ... 94
On the AVSDT-Coloring of $S_m + W_n$
BY ZHONGFU ZHANG, ENQIANG ZHU, BAOGEN XU ET AL ... 105
Actions of Multi-groups on Finite Sets
BY LINFAN MAO ... 111

ISBN 1-59973-076-6
53995>
9 781599 730769